Lecture 14
- how to design an observer? -

The observer

We consider the following system

$$\frac{dx}{dt} = Ax + Bu, \quad y = Cx$$

Attempt to determine the state by simulating the equations with the input

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu$$

Consider the estimation error $\tilde{x} = x - \hat{x}$.

The error dynamics are given by

$$\frac{d\tilde{x}}{dt} = A\tilde{x}$$

If matrix A has all its eigenvalues in the left half-plane, the error $\tilde{x}$ will go to zero, that is the error dynamics converges

The observer

Consider the observer

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + L(y - C\hat{x})$$

- Added the term $L(y - C\tilde{x})$
- The term is proportional to the difference between:
  - the observed output y
  - the output predicted by the observer $C\hat{x}$
- The error dynamics are driven by

$$\frac{d\tilde{x}}{dt} = (A - LC)\tilde{x}$$

If the matrix L can be chosen in such a way that the matrix $A - LC$ has eigenvalues with negative real parts, the error $\tilde{x}$ will go to zero

The convergence rate is determined by an appropriate selection of the eigenvalues

Theorem

Consider the system given by

$$\frac{dx}{dt} = Ax + Bu, \quad y = Cx$$

with one input and one output. Let $\lambda(s) = s^n + a_1 s^n - 1 + \cdots + a_{n-1} s + a_n$ be the characteristic polynomial for A.

If the system is observable, then the dynamical system

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + L(y - C\hat{x})$$

is an observer for the system.

Theorem (cont.)

The vector L can be chosen as

$$L = W_o^{-1}\tilde{W}_o \begin{bmatrix} p_1 - a_1 \\ p_2 - a_2 \\ \vdots \\ p_n - a_n \end{bmatrix}$$

where

$$W_o = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \qquad \tilde{W}_o = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ a_1 & 1 & 0 & \cdots & 0 & 0 \\ a_2 & a_1 & 1 & & 0 & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ a_{n-2} & a_{n-3} & a_{n-4} & & 1 & 0 \\ a_{n-1} & a_{n-2} & a_{n-3} & \ldots & a_1 & 1 \end{bmatrix}^{-1}$$

The resulting observer error $\tilde{x} = x - \hat{x}$ is governed by a differential equation having the characteristic polynomial $p(s) = s_n + p_1 s^{n-1} + \cdots + p^n$.