Lecture 9

- how to approximate around an equilibrium? -

Review: How to find a Lyapunov function?

- For linear systems $\dot{x} = Ax$
- Let $V(x) = x^\top P x$
- V is positive definite $\iff$ P is a positive definite matrix ($P > 0$)
- Compute its derivative:

$$\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dV}{dx} = x^\top \overbrace{(A^\top P + PA)}^{Q < 0} x$$

- Let $Q = -I \in \mathbb{R}^{n \times n}$
- Solve for P in $Q = A^\top P + PA < 0$
- Verify whether $x^\top P x > 0$ for all $x \neq 0$

Example

$$\frac{dx_1}{dt} = -ax_1$$

$$\frac{dx_2}{dt} = -bx_1 - cx_2$$

$$A = \begin{bmatrix} -a & 0 \\ -b & -c \end{bmatrix} \qquad P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

Define Q

$$A^T P + PA = -I$$

$$\begin{bmatrix} -a & -b \\ 0 & -c \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} -a & 0 \\ -b & -c \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Solving for P

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} \frac{b^2 + ac + c^2}{2a^2c + 2ac^2} & -\frac{b}{2c(a+c)} \\ -\frac{b}{2c(a+c)} & \frac{1}{2c} \end{bmatrix}$$

Verify that $P > 0$

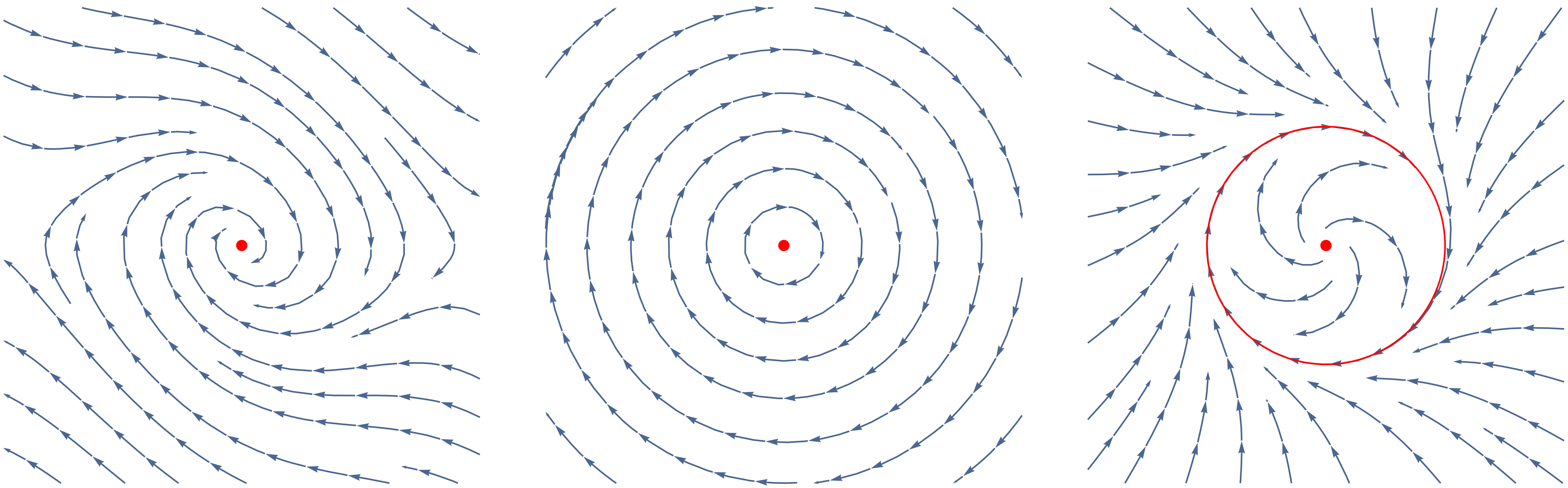
Resulting Lyapunov function

$$V(x) = \frac{b^2 + ac + c^2}{2a^2c + 2ac^2} x_1^2 - \frac{b}{c(a+c)} x_1 x_2 + \frac{1}{2c} x_2^2$$

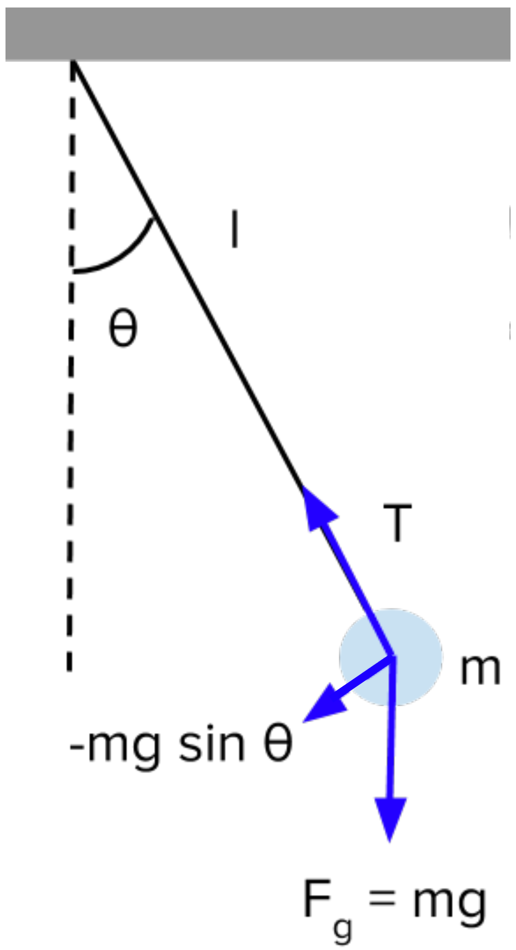
Theorem (Krasovskii-Lasalle). Let $V : \mathbb{R}^n \mapsto \mathbb{R}$ be a locally positive definite function such that, on the compact set $\Omega_r = \{x \in \mathbb{R} : V(x) \leq r\}$, we have $\dot{V}(x) \leq 0$. Define

$$S = \{x \in \Omega_r : \dot{V}(x) = 0\}.$$

As $t \to \infty$, any trajectory starting inside Ω_r converges to the largest invariant set inside S. That is, the ω limit set of any such trajectory is contained inside the largest infant set in S. In particular, if S contains no invariant set except the equilibrium $x_e = 0$, then the origin is asymptotical stable.



Example



Consider the system

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{g}{\ell}\sin x_1 - \frac{k}{m}x_2 \end{bmatrix}$$

And the Lyapunov function

$$V(x) = \frac{g}{\ell}(1 - \cos x_1) + \frac{1}{2}x_2^2$$

Then

$$\dot{V}(x) = \frac{g}{\ell}\sin x_1 \dot{x}_1 + x_2\dot{x}_2$$

Define largest invariant set for $-\pi < x_1 = \theta < \pi$

$$S = \{(x_1, x_2) : \dot{V}(x) = 0\} \implies \text{does not contain any invariant set, except } \mathbf{x} = 0$$

When does the derivative equals zero?

$$0 = \frac{g}{\ell}\sin x_1 \dot{x}_1 + x_2\dot{x}_2$$

$$= -\frac{g}{\ell}x_2^2 \implies x_2 = 0$$

If $x_2 = 0$, then $\dot{x}_2 = 0$ and

$$0 = -\frac{g}{\ell}\sin x_1 \implies x_1 = 0$$

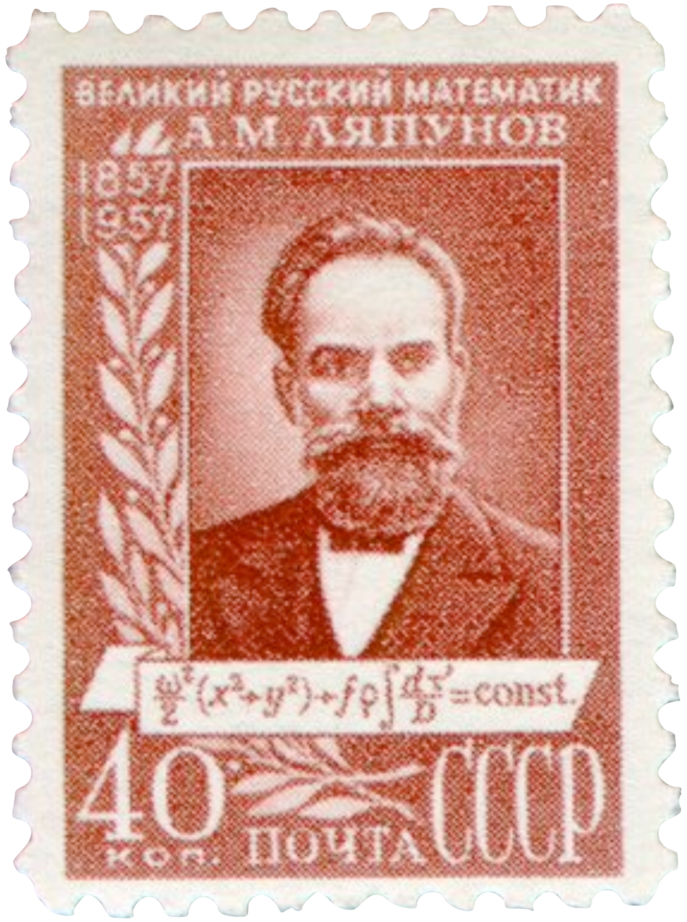
Lyapunov theory (discrete-time version)

Theorem

Consider a nonlinear discrete-time system with dynamics $x_{k+1} = f(x_k)$ and equilibrium point $x_e = 0$. Suppose there exists a smooth, positive definite function $V : \mathbb{R}^n \mapsto \mathbb{R}$ such that

$$V(x_{k+1}) - V(x_k) < 0 \text{ and } V(0) = 0$$

Then $x_e = 0$ is (locally) asymptotically stable.



Today

- Linearization of a nonlinear system around an equilibrium point
- Stability of linear systems
- Lyapunov's Indirect Method

Next class

- Define reachability of a system
- Give test for reachability of linear systems
- Describe state feedback for linear systems

Linearization

Consider a single-input, single-output nonlinear system

$$\frac{dx}{dt} = f(x,u), \qquad x \in \mathbb{R}^n, u \in \mathbb{R}$$
$$y = h(x,u), \qquad y \in \mathbb{R}$$

with an equilibrium point at $x = x_e$, $u = u_e$.

Define a new set of state variables z, input v and output w as:

$$z = x - x_e, \quad v = u - u_e, \quad w = y - h\left(x_e, u_e\right)$$

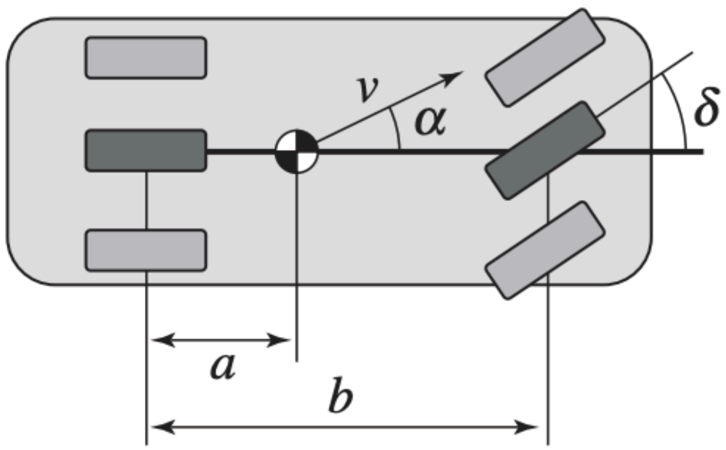
The Jacobian linearization of the nonlinear system is

$$\frac{dz}{dt} = Az + Bv, \quad w = Cz + Dv$$

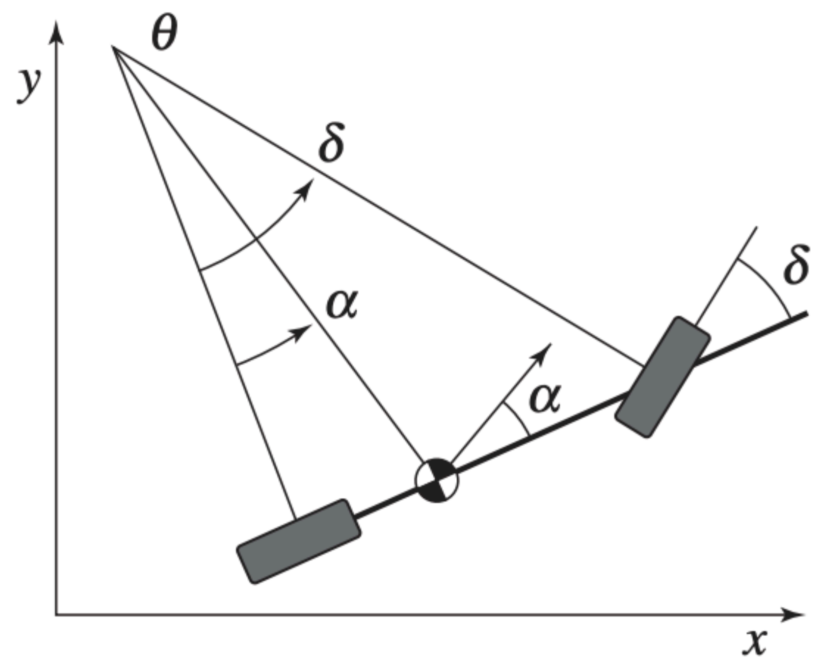
where

$$A = \left.\frac{\partial f}{\partial x}\right|_{\left(x_e,u_e\right)}, \quad B = \left.\frac{\partial f}{\partial u}\right|_{\left(x_e,u_e\right)}, \quad C = \left.\frac{\partial h}{\partial x}\right|_{\left(x_e,u_e\right)}, \quad D = \left.\frac{\partial h}{\partial u}\right|_{\left(x_e,u_e\right)}$$

Example - vehicle steering dynamics (bicycle model)



overhead view of a vehicle with 4 wheels



Approximation by a single front wheel and a single rear whee

δ : steering angle

α : angle of the velocity at the center of mass (relative the length axis)

The position of the vehicle is given by (x, y) and the orientation (heading) by θ

The nonlinear equations of motion for the system are given

$$\frac{\mathrm{d}}{\mathrm{dt}}\begin{bmatrix} x \\ y \\ \theta \end{bmatrix} = \begin{bmatrix} v\cos(\alpha(\delta)+\theta) \\ v\sin(\alpha(\delta)+\theta) \\ \frac{v_0}{b}\tan\delta \end{bmatrix}, \qquad \alpha(\delta) = \arctan\left(\frac{a\tan\delta}{b}\right)$$

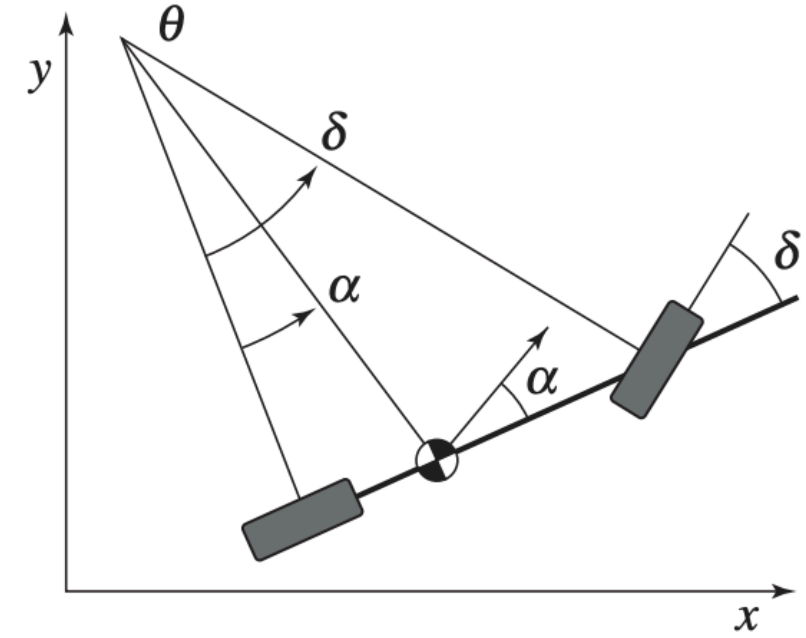
where v_0 is the velocity of the rear wheel, a the stance from the rear wheels to the center of mass, and b is the distance between the front and rear wheels.

Example - vehicle steering dynamics (bicycle model)

- Motion of the vehicle about a straight-line:

$$\theta = \theta_0 \text{ with fixed velocity } v_0 \neq 0$$

- set $\dot{\theta} = 0 \implies \delta = 0$ (steering wheel is straight)
- Also implies $\alpha(\delta) = 0$
- The motion in the xy direction is by definition not at equilibrium



$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ \theta \end{bmatrix} = \begin{bmatrix} v\cos(\alpha(\delta)+\theta) \\ v\sin(\alpha(\delta)+\theta) \\ \frac{v_0}{b}\tan\delta \end{bmatrix}$$

$$\alpha(\delta) = \arctan\left(\frac{a\tan\delta}{b}\right)$$

Example - vehicle steering dynamics (bicycle model)

- Lateral deviation of the vehicle from a straight line:

$$\theta_e = 0$$

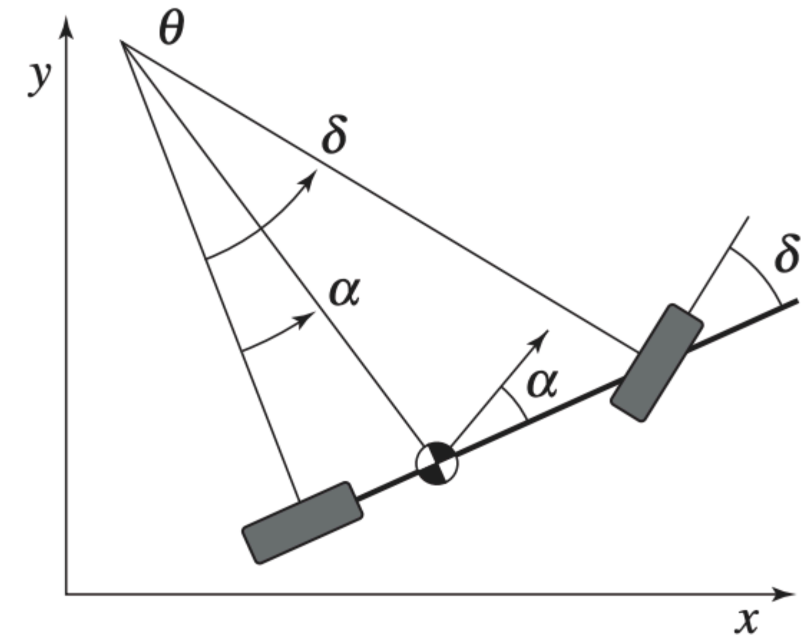
- Vehicle driving along the x axis
- Focus on the motion in the y and θ directions, ignore x direction
- Let the state $x = (x_1, x_2) = (y, \theta)$ and $y = \delta$
- The system can we rewritten in standard form:

$$f(x, u) = \begin{bmatrix} v \sin\left(\alpha(u) + x_2\right) \\ \frac{v_0}{b} \tan u \end{bmatrix}$$

$$\alpha(u) = \arctan\left(\frac{a \tan u}{b}\right)$$

$$h(x, u) = x_1$$

The equilibrium of interest is given by
$x = (0, 0)$ and $u = 0$



$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ \theta \end{bmatrix} = \begin{bmatrix} v \cos(\alpha(\delta) + \theta) \\ v \sin(\alpha(\delta) + \theta) \\ \frac{v_0}{b} \tan \delta \end{bmatrix}$$

$$\alpha(\delta) = \arctan\left(\frac{a \tan \delta}{b}\right)$$

Example - linearized model

Original dynamics

$$f(x,u) = \begin{bmatrix} v\sin\big(\alpha(u) + x_2\big) \\ \frac{v_0}{b}\tan u \end{bmatrix}, \qquad \alpha(u) = \arctan\left(\frac{a\tan u}{b}\right)$$

Linearized dynamics

$$A = \frac{\partial f}{\partial x}\bigg|_{\substack{x=0\\u=0}} = \begin{bmatrix} 0 & v_0 \\ 0 & 0 \end{bmatrix} \qquad B = \frac{\partial f}{\partial u}\bigg|_{\substack{x=0\\u=0}} = \begin{bmatrix} \frac{av_0}{b} \\ \frac{v_0}{b} \end{bmatrix} \qquad \Longrightarrow \qquad \frac{dz}{dt} = Az + Bw$$

Original output

$$h(x,u) = x_1$$

Linearized output

$$C = \frac{\partial h}{\partial x}\bigg|_{\substack{x=0\\u=0}} = [1 \quad 0] \qquad D = \frac{\partial h}{\partial u}\bigg|_{\substack{x=0\\u=0}} = 0 \qquad \Longrightarrow \qquad y = Cz + Dw$$