Lecture 8

- do trajectories converge to invariant sets? -

Today

- Krasovskii-Lasalle Invariance Principle

Next class

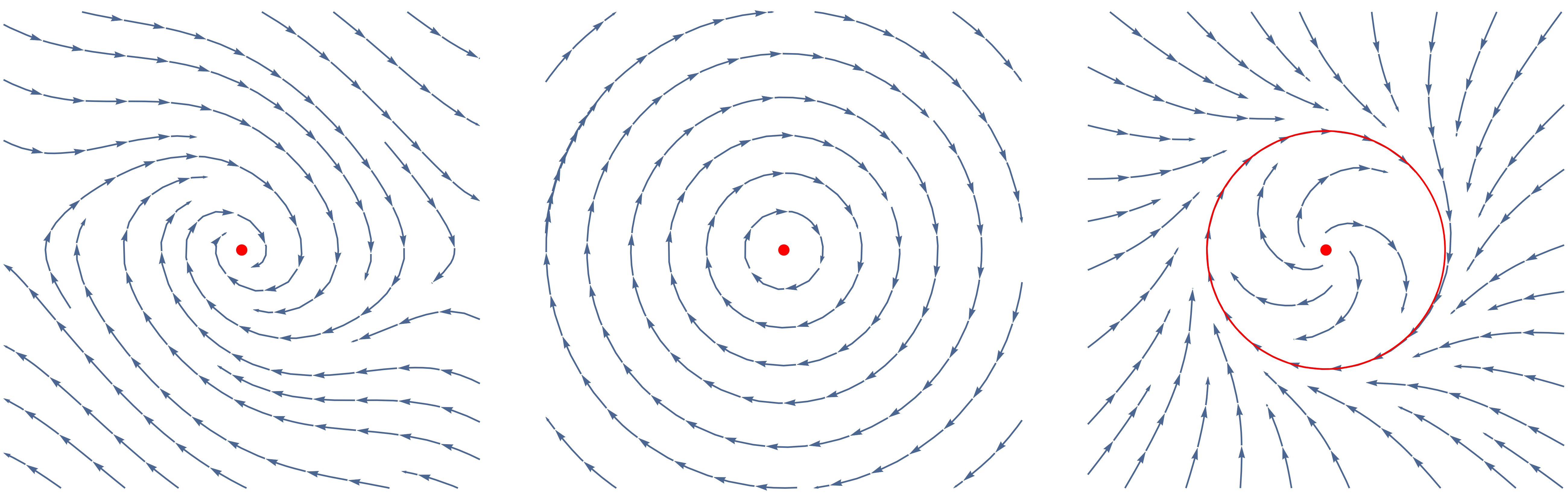
- Compute linearization of a nonlinear system around an equilibrium point

Definition (ω limit set). The ω limit set of a trajectory $x(\cdot\,; x_0, t_0)$ is the set of all points $z \in \mathbb{R}$ such that there exists a strictly increasing sequence of time t_n that satisfies

$$\lim_{n \to \infty} t_n = \infty$$

and as $n \to \infty$

$$x(t_n; x_0, t_0) = z$$



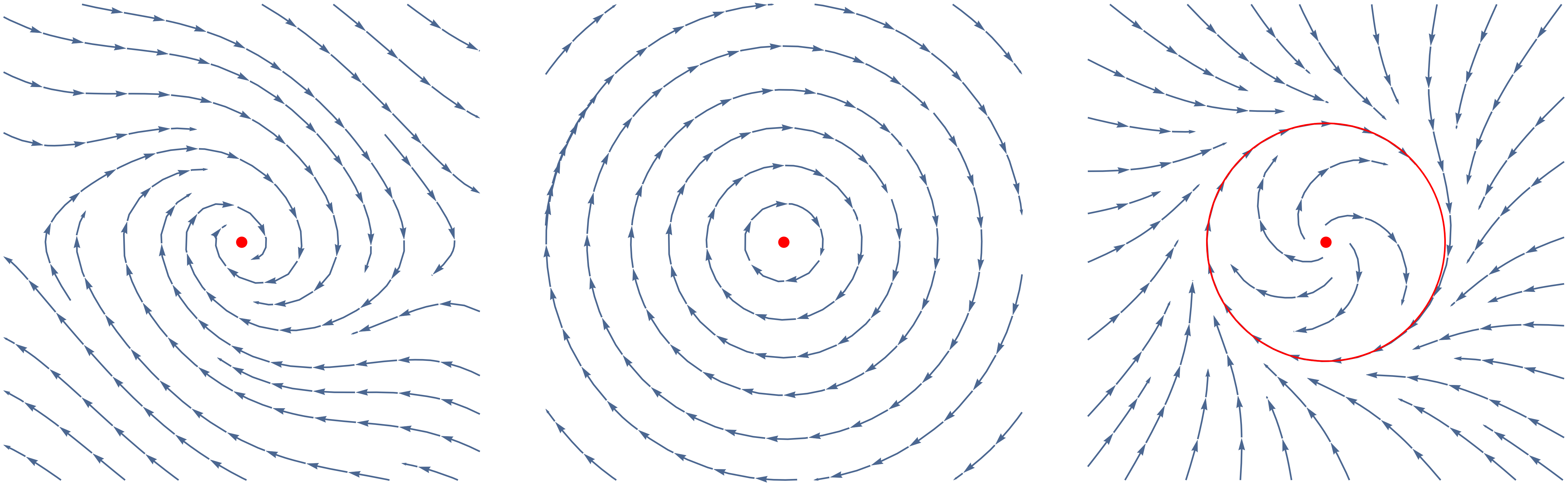
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\sin x_1 - 0.5 x_2$$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1$$

$$\dot{x}_1 = 4x_2 - x_1\left(x_1^2 + x_2^2 - 4\right)$$
$$\dot{x}_2 = -4x_1 - x_2\left(x_1^2 + x_2^2 - 4\right)$$

Definition (invariant set). The set $M \subset \mathbb{R}^n$ is said to be an *invariant set* if for all $y \in M$ and $t_0 \geq 0$, we have

$$x(t; y, t_0) \in M \text{ for all } t \geq t_0.$$



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Theorem (Krasovskii-Lasalle). Let $V : \mathbb{R}^n \mapsto \mathbb{R}$ be a locally positive definite function such that, on the compact set $\Omega_r = \{x \in \mathbb{R} : V(x) \leq r\}$, we have $\dot{V}(x) \leq 0$. Define

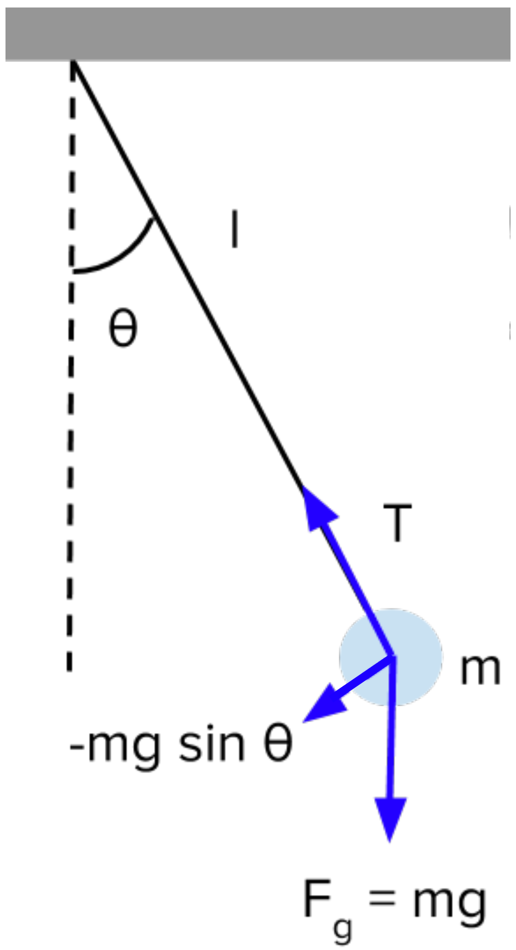
$$S = \{x \in \Omega_r : \dot{V}(x) = 0\}.$$

As $t \to \infty$, any trajectory starting inside Ω_r converges to the largest invariant set inside S. That is, the ω limit set of any such trajectory is contained inside the largest infant set in S. In particular, if S contains no invariant set except the equilibrium $x_e = 0$, then the origin is asymptotical stable.

Remarks:

- The word "largest" refers to the union of all invariant sets within S
- If r is infinity, then the origin is globally asymptotically stable

Example



Consider the system

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{g}{\ell}\sin x_1 - \frac{k}{m}x_2 \end{bmatrix}$$

And the Lyapunov function

$$V(x) = \frac{g}{\ell}(1 - \cos x_1) + \frac{1}{2}x_2^2$$

Then

$$\dot{V}(x) = \frac{g}{\ell}\sin x_1 \dot{x}_1 + x_2 \dot{x}_2$$

Define largest invariant set for $-\pi < x_1 = \theta < \pi$

$$S = \{(x_1, x_2) : \dot{V}(x) = 0\} \implies \text{does not contain any invariant set, except } x = 0$$

When does the derivative equals zero?

$$0 = \frac{g}{\ell}\sin x_1 \dot{x}_1 + x_2 \dot{x}_2$$

$$= -\frac{g}{\ell}x_2^2 \implies x_2 = 0$$

If $x_2 = 0$, then $\dot{x}_2 = 0$ and

$$0 = -\frac{g}{\ell}\sin x_1 \implies x_1 = 0$$