Lecture 7

- how to find Lyapunov functions? -

What we already know...

Aleksandr Mikhailovich Lyapunov

Born	June 6, 1857 Yaroslavl, Imperial Russia
Died	November 3, 1918 (aged 61)
Residence	Russia
Nationality	Russian
Fields	Applied mathematics
Institutions	Saint Petersburg State University Russian Academy of Sciences Kharkov University
Alma mater	Saint Petersburg State University
Known for	Lyapunov function

Review (definition of stability)

An equilibrium point $x_e = 0$ is locally asymptotically stable, if the following two conditions are satisfied:

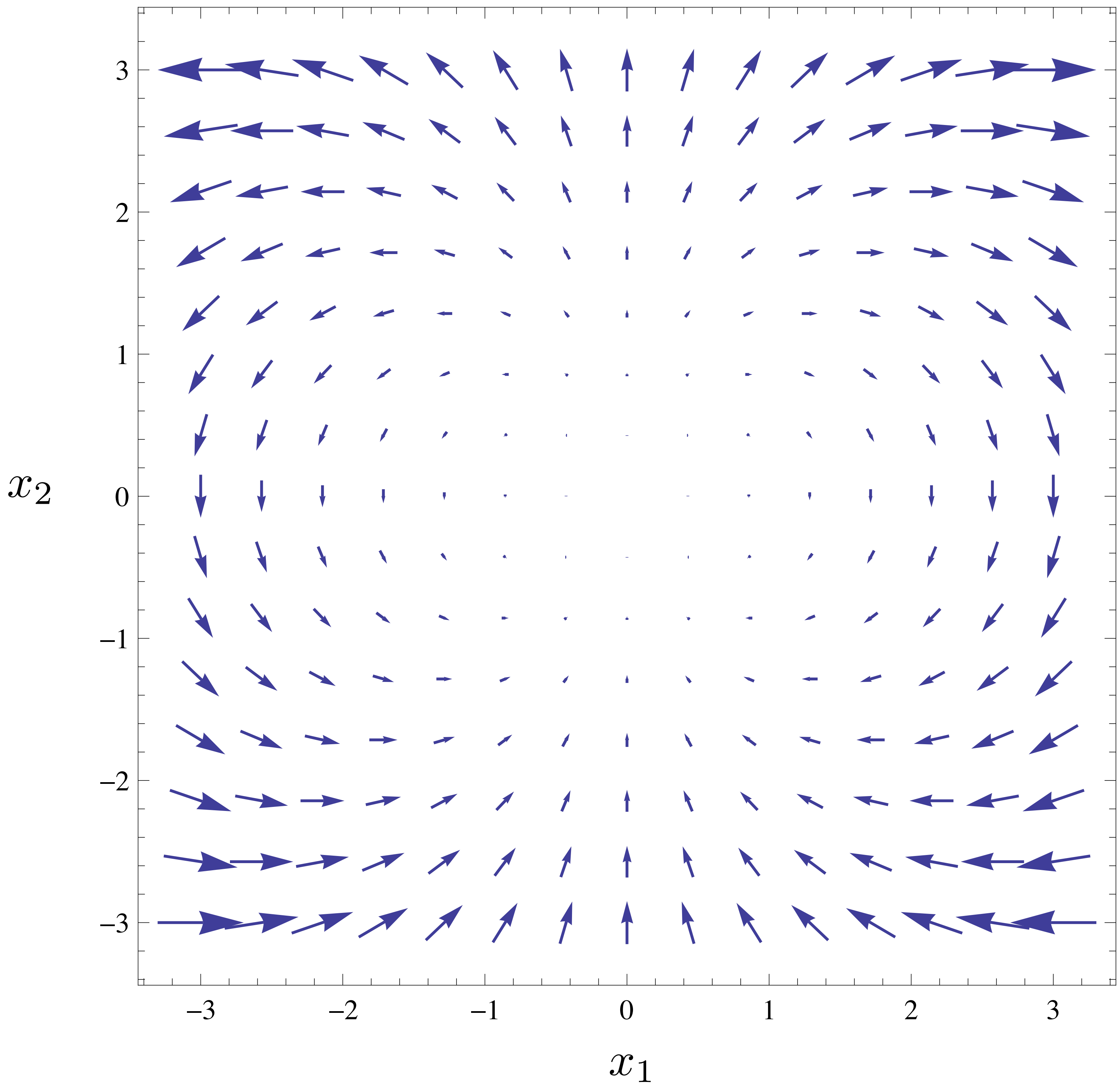
1. For every $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that:

$$\text{If } \|x(0) - x_e\| < \delta, \text{ then } \|x(t) - x_e\| < \varepsilon, \text{ for every } t \geq 0.$$

2. There exists a $\delta > 0$ such that:

$$\text{If } \|x(0) - x_e\| < \delta, \text{ then } \lim_{t \to \infty} \|x(t) - x_e\| = 0$$

Why capture the notion of stability?



$$\dot{x}_1 = 2x_1x_2$$
$$\dot{x}_2 = x_2^2 - x_1^2$$

Review (definition of positive definite)

Definition: A continuous function $V : \mathbb{R}^n \mapsto \mathbb{R}$ is (locally) positive definite, if for some $\delta > 0$:

1. $V(0) = 0$
2. $V(x) > 0$ for all $x \neq 0$ and $\|x\| < \delta$

The function V is (locally) positive semi-definite if for some $\delta > 0$:

1. $V(0) = 0$
2. $V(x) \geq 0$ for all $\|x\| < \delta$

Review (Lyapunov theory)

Theorem
Let V be a non-negative function on $\mathbb{R}^n$ and $\dot{V}$ represent the time derivative of V along the trajectories of the system dynamics

$$\dot{V} = \frac{\partial V}{\partial x}\frac{dx}{dt} = \frac{\partial V}{\partial x} f(x)$$

Let $B_r(0) = \{x \in \mathbb{R}^n : |x| < r\}$ be a ball of radius r around the origin. If there exists $r > 0$ such that V is positive definite and

- The derivative $\dot{V}$ is negative semi-definite for all $x \in B_r$, then $x_e = 0$ is stable in the sense of Lyapunov
- The derivative $\dot{V}$ is negative definite for all $x \in B_r$, then $x_e = 0$ is locally asymptotically stable
- The derivative $\dot{V} = -kV$ for all $x \in B_r$, then $x_e = 0$ is locally exponentially stable
- The derivative $\dot{V}$ is positive definite for all $x \in B_r$, then $x_e = 0$ is unstable

One cannot really argue with
a mathematical theorem

Stephen Hawking

When we prove a mathematical result,
we are demonstrating what follows
from our assumptions

How to find a Lyapunov function?

- Lyapunov function are generally hard to define
- Just because we cannot prove that something is true (find proper Lyapunov function), does not mean it is false
- In general (linear + non-linear systems):
  - Before guess (before 2000)
  - Now use SOS-tools (Matlab toolbox) for polynomial systems with known coefficients

How about for linear systems?

Today

- Lyapunov functions for linear systems

How to find a Lyapunov function?

- For linear systems $\dot{x} = Ax$
- Let $V(x) = x^{\top}Px$
- V is positive definite $\iff$ P is a positive definite matrix $(P > 0)$
- Compute its derivative:

$$\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dV}{dx} = x^{\top}\overbrace{(A^{\top}P + PA)}^{Q<0}x$$

- Let $Q = -I \in \mathbb{R}^{n \times n}$
- Solve for P in $Q = A^{\top}P + PA < 0$
- Verify whether $x^{\top}Px > 0$ for all $x \neq 0$

Example

$$\frac{dx_1}{dt} = -ax_1$$

$$\frac{dx_2}{dt} = -bx_1 - cx_2$$

$$A = \begin{bmatrix} -a & 0 \\ -b & -c \end{bmatrix} \qquad P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

Define Q

$$A^T P + PA = -I$$

$$\begin{bmatrix} -a & -b \\ 0 & -c \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} -a & 0 \\ -b & -c \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Solving for P

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} \frac{b^2 + ac + c^2}{2a^2c + 2ac^2} & -\frac{b}{2c(a+c)} \\ -\frac{b}{2c(a+c)} & \frac{1}{2c} \end{bmatrix}$$

Verify that $P > 0$

Resulting Lyapunov function

$$V(x) = \frac{b^2 + ac + c^2}{2a^2c + 2ac^2} x_1^2 - \frac{b}{c(a+c)} x_1 x_2 + \frac{1}{2c} x_2^2$$

How to determine positive-definiteness

- A is Hermitian (or symmetric in the case of real matrices) matrix M is positive definite if:
  - All the diagonal entries are positive
  - Each diagonal entry is greater than the sum of the absolute values of all other entries in the same row.

Sufficient but not necessary!

Necessary and sufficient criteria

- A is Hermitian (or symmetric in the case of real matrices) matrix M is positive definite if and only if:
  - All the following matrices have a positive determinant:
    - the upper left 1-by-1 corner of M,
    - the upper left 2-by-2 corner of M,
    - ...
    - M itself.

    Sylvester's criterion
  - All the eigenvalues of M are positive.