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Stability of the Jackson-Rogers model

Diego Ruiz[1,2] and Jorge Finke[3]
[1]Department of Mathematics, Universidad del Cauca, Popayán, Colombia.
[2]School of Systems Engineering and Computer Science, Universidad del Valle, Cali, Colombia.
[3]Department of Electrical Engineering and Computer Science, Pontificia Universidad Javeriana, Cali, Colombia.
dfruiz@unicauca.edu.co, finke@ieee.org

Abstract— **Network formation models explain the dynamics of the structure of connections using mechanisms that operate under different principles for establishing and removing edges. The Jackson-Rogers model is a generic framework that applies the principle of triadic closure to growing networks. Past work describes the asymptotic behavior of the degree distribution based on a continuous-time approximation. Here, we introduce a discrete-time approach that provides a more accurate fit of the dynamics of the in-degree distribution of the Jackson-Rogers model. Furthermore, we characterize the limit distribution and the expected value of the average degree as equilibria, and prove that both equilibria are asymptotically stable.**

I. Introduction

Power laws describe probability distributions of the degrees of nodes in social, biological, and information networks [1], [2]. To gain insight into the underlying dynamics, a common approach has been to derive mathematical representations of mechanisms responsible for such distributions. Preferential attachment refers to a class of mechanisms, in which the probability that a new node connects to existing nodes is proportional to their degree [3]. Based on preferential attachment, the work in [4] introduces the Albert-Barabasi model, which generates power law degree distributions with a scaling exponent $\beta = 3$. Extensions to the Albert-Barabasi model include local and non-linear variations to the original attachment rule [5]–[7]. Yet, none of these preferential attachment models recreate the clustering properties of many empirical networks [8].

To address this limitation, the work in [9] introduces the Jackson-Rogers model based on the principle of triadic closure. The model uses three mechanisms to explain the formation of clustering. First, a new node selects r target nodes and connects to each of them with probability p_r. Second, the new node establishes additional edges to c neighbors of the target nodes with probability p_c. And third, a total of n nodes establish edges to the new node. Using a continuos-time approximation, the authors in [9] show that for large networks, the in-degree distribution of the Jackson-Rogers model converges to a power law with $\beta = 2 + \frac{rp_r}{cp_c}$. Furthermore, as $\frac{rp_r}{cp_c} \to 0$, the Jackson-Rogers model can be viewed as a network model with indirect preferential attachment. In such cases, triadic closure increases the probability that the edges to the neighbors of the target nodes are directed to nodes with a high in-degree.

There are a number of variations to the original mechanisms of the Jackson-Rogers model [10], [11]. The work in [11] considers the effects of reciprocal edges on power law distributions. It shows that as the network grows, reciprocity and clustering can be characterized as equilibria that are asymptotically stable. Stability of these equilibria means that for all initial network configurations, the two properties remain close to their convergence value for all time. To the best of our knowledge, the stability of the degree distribution of the Jackson-Rogers model has not been studied.

The contributions of this paper are the following. First, we use a discrete-time approach to characterize the probability that a randomly selected node has a particular in-degree (Theorem 1). Second, we show that the dynamics of the in-degree distribution converge to stationary distributions (Corollary 1). This corollary holds for the Jackson-Rogers model and for any network model where the number of edges grows linearly with time (e.g., for the Albert-Barabasi model). Unlike [12], we allow new nodes to have a non-zero in-degree (i.e., the model satisfies $n \neq 0$). Third, we characterize the complementary cumulative in-degree distribution as an invariant set and show that this set is asymptotically stable. Finally, we show that the value of the expected average in-degree is also asymptotically stable.

The remainder of this paper is organized as follows. Section II characterizes the in-degree distribution of the model. Section III presents the stability properties. Section IV presents simulation results. Section V draws some conclusions and future work.

II. Asymptotic behavior of the in-degree distribution

Let I denote an index set of non-negative integer numbers. Consider a sequence $\mathcal{G} = \{\mathcal{G}_i\}_{i\in I}$, where $\mathcal{G}_t = (V_t, E_t)$ represents a directed network at time $t \geq 0$ with set of nodes V_t and set of edges $E_t \subseteq V_t \times V_t$. Let K_t denote a random variable that characterizes the in-degree of a node selected uniformly at random at time t. Moreover, let $P_t(k) = P(K_t = k)$ denote the probability that a realization of K_t equals k. The cumulative distribution function of the in-degree of the nodes of $\mathcal{G}_t$ is denoted by $F_t(k) = P(K_t < k) = \sum_{x<k} P_t(x)$. Moreover, $F_t^c(k) = P(K_t \geq k) = 1 - F_t(k)$ denotes the complementary cumulative distribution function.

The Jackson-Rogers model uses three mechanisms to establish new edges [9]:

M1 *Random attachment:* A new node chooses r target nodes, selected uniformly at random from the set of nodes in V_{t-1}, and connects independently to each of them with probability p_r.

M2 *Triadic formation:* Of the union of the outgoing neighbors of all target nodes, the new node chooses c nodes, selected uniformly at random, and connects independently to each of them with probability p_c.

M3 *Network response:* A total of n nodes, selected uniformly at random from the set of nodes V_{t-1}, connect to the new node.

Mechanism M1 indicates that the new node tries to connect to r target nodes. The neighbors of these nodes are the support set for the selection process of mechanism M2. Note that mechanism M3 guarantees that the in-degree of the new node is equal to n.

To ensure a well-defined network formation process, consider the following assumptions:

A1 The number of nodes in the initial network satisfies $n_0 \geq r + c$.

A2 Mechanism M3 satisfies $0 \leq n \leq n_0$.

Assumption A1 guarantees that the node added at $t = 1$ can connect to up to $r + c$ nodes. Assumption A2 guarantees that the number of edges established due to the response of the network is at most n_0.

Let R_t and C_t denote random variables that characterize the number of edges established by a new node due to M1 and M2. Each mechanism follows a Bernoulli process, so R_t and C_t are binomially distributed variables with expected values $\mathrm{E}[R_t] = rp_r$ and $\mathrm{E}[C_t] = cp_c$. Let $M_t = R_t + C_t$ represent a random variable that characterizes the total number of edges established by the new node through both mechanisms. Because R_t and C_t are independent, $\mathrm{E}[M_t] = \mathrm{E}[R_t] + \mathrm{E}[C_t] = rp_r + cp_c$. Let $m = rp_r + cp_c + n$.

Next, let $\pi_t(k)$ denote the probability that the new node connects to a node of in-degree k. We know that

$$\pi_t(k) = \frac{rp_r}{n_{t-1}} + \frac{kcp_c}{n_{t-1}m} \tag{1}$$

where $n_t = |V_t|$. The first term of eq. (1) represents the probability that the new node connects to a node of in-degree k through mechanism M1. The second term represents the probability that the new node connects to a node of in-degree k through mechanism M2 [9], [13].

We use the notion of asymptotic equivalence between two real sequences to guarantee the existence of $\lim_{t\to\infty} P_t(k)$ for $k \geq n$ and to specify the limit distribution of the in-degree of the network.

Lemma 1: Let $\{s_t\}$ and $\{u_t\}$ be two equivalent sequences of positive real numbers, denoted by $s_t \sim u_t$. That is, $\lim_{t\to\infty} s_t/u_t = 1$. If $\lim_{t\to\infty} u_t = L < \infty$, then $\lim_{t\to\infty} s_t = L$. Moreover, if $v_t \sim w_t$, then $s_t + v_t \sim u_t + w_t$.

Proof: First, if $s_t \sim u_t$, then for all $\varepsilon > 0$ there exists $T_1 \in \mathbb{N}$ such that for all $t > T_1$, $\left|\frac{s_t}{u_t} - 1\right| < \varepsilon$. If $\lim_{t\to\infty} u_t = L$, then for all $\varepsilon > 0$ there exists $T_2 \in \mathbb{N}$ such that for all $t > T_2$, $|u_t - L| < \varepsilon$. So

$$\begin{aligned} |s_t - L| &\leq |s_t - u_t| + |u_t - L| \\ &< |s_t - u_t| + \varepsilon \\ &= |u_t| \left|\frac{s_t}{u_t} - 1\right| + \varepsilon \\ &< (\varepsilon + L)\varepsilon + \varepsilon = \varepsilon' \end{aligned}$$

Therefore, for all $\varepsilon' > 0$, there exists $T = \max\{T_1, T_2\}$ such that for all $t > T$, $|s_t - L| < \varepsilon'$.

Second, because $\{s_t\}, \{u_t\}, \{v_t\}$ and $\{w_t\}$ are sequences of positive real numbers, if $\frac{s_t}{u_t} < \frac{v_t}{w_t}$, then

$$\frac{s_t}{u_t} < \frac{s_t + v_t}{u_t + w_t} < \frac{v_t}{w_t}$$

Since $s_t \sim u_t$ and $v_t \sim w_t$, applying the Squeeze Theorem, we have

$$\lim_{t\to\infty} \frac{s_t + v_t}{u_t + w_t} = 1$$

That is, $s_t + v_t \sim u_t + w_t$. The same reasoning applies for $\frac{v_t}{w_t} \leq \frac{s_t}{u_t}$. ∎

The following theorem guarantees that as the network grows the probability of the in-degree distribution of the Jackson-Rogers model converges.

Theorem 1: As t tends to infinity, the limit of $P_t(k)$ exists for all $k \geq n$.

Proof: First, we determine a recursive expression for $P_t(k)$ for all $k \geq n$. According to eq. (1), we know that the expected number of nodes of in-degree k is

$$\pi_t(k) n_{t-1} P_{t-1}(k) = \left(rm_r + \frac{kcp_c}{m}\right) P_{t-1}(k) \tag{2}$$

Using eq. (2), the expected number of nodes of in-degree $k > n$ is

$$\begin{aligned} n_t P_t(k) &= (n_{t-1} - \pi_t(k) n_{t-1}) P_{t-1}(k) \\ &\quad + \pi_t(k-1) n_{t-1} P_{t-1}(k-1) \\ &= \left(n_{t-1} - rp_r - \frac{kcp_c}{m}\right) P_{t-1}(k) \\ &\quad + \left(rp_r + \frac{(k-1)cp_c}{m}\right) P_{t-1}(k-1) \end{aligned} \tag{3}$$

That is, at time t, the expected number of nodes of in-degree $k > n$ is equal to the difference between the expected number of nodes of in-degree k and the expected number of nodes of in-degree k selected at time $t-1$ by mechanisms M1 or M2, plus the expected number of nodes of in-degree $k-1$ that establish an edge with the new node.

Now, because n nodes establish an edge to the new node, the expected number of nodes of in-degree $k = n$ is

$$\begin{aligned} n_t P_t(n) &= n_{t-1} P_{t-1}(n) - \pi_t(n) n_{t-1} P_{t-1}(n) + 1 \\ &= \left(n_{t-1} - rp_r - \frac{ncp_c}{m}\right) P_{t-1}(n) + 1 \end{aligned} \tag{4}$$

The first term represents the difference between the expected number of nodes of in-degree n at time $t-1$ and the expected number of nodes of in-degree n that establish at time t an edge with the new node. The number 1 accounts for the new node attaching to the network with in-degree n.

Second, using eqs. (3) and (4), we proceed by induction over k to guarantee the existence of $\lim_{t\to\infty} P_t(k)$. Consider $k = n$ as the base case. Using eq. (4), note that $P_t(n)$ can be expressed using the recurrence

$$P_t(n) = \frac{1}{n_t}\left(n_{t-1} - rp_r - \frac{ncp_c}{m}\right)P_{t-1}(n) + \frac{1}{n_t}$$

It can be shown by induction that for $b = 1 + rp_r + \frac{ncp_c}{m}$ and $t \geq 0$

$$P_t(n) = \frac{(bP_0(n) - 1)\Gamma(n_1)\Gamma(n_{t+1} + b)}{b\Gamma(n_1 - b)\Gamma(n_{t+1})} + \frac{1}{b} \qquad (5)$$

where $\Gamma(\cdot)$ denotes the gamma function. According to assumption A1, eq. (5) is well-defined. Because $b > 0$, note that

$$\Gamma(n_{t+1} - \lfloor b \rfloor) \leq \Gamma(n_{t+1} - b) \leq \Gamma(n_t)$$

where $\lfloor \cdot \rfloor$ denotes the floor function. Applying the Squeeze Theorem, we have

$$\lim_{t\to\infty} \frac{\Gamma(n_{t+1} - b)}{\Gamma(n_{t+1})} = 0$$

and

$$\lim_{t\to\infty} P_t(n) = \frac{m}{m(1 + rp_r) + ncp_c}$$

Now, assume that $\lim_{t\to\infty} P_t(k)$ exists for all $k > n$. Using eq. (3), we get

$$\begin{aligned} n_t P_t(k+1) &= \left(rp_r + \frac{kcp_c}{m}\right) P_{t-1}(k) \\ &+ \left(n_{t-1} - rp_r - \frac{(k+1)cp_c}{m}\right) P_{t-1}(k+1) \qquad (6) \end{aligned}$$

Because the number of edges that new nodes establish to existing nodes is bounded by $r + c$, for $k \geq n$ and a large enough t, we know that $P_{t-1}(k) \sim P_t(k)$ and $P_{t-1}(k+1) \sim P_t(k+1)$. Using Lemma 1 and eq. (6), for a large enough t

$$\left(1 + rp_r + \frac{(k+1)cp_c}{m}\right) P_t(k+1) \sim \left(rp_r + \frac{kcp_c}{m}\right) P_t(k)$$

Based on the inductive hypothesis, we know that $\lim_{t\to\infty} P_t(k)$ exists, so

$$\lim_{t\to\infty} P_t(k+1) = \frac{rp_r + \frac{kcp_c}{m}}{1 + rp_r + \frac{(k+1)cp_c}{m}} \lim_{t\to\infty} P_t(k)$$

Therefore, $\lim_{t\to\infty} P_t(k)$ exists for all $k \geq n$. ■

We use Theorem 1 to characterize the in-degree distribution of the network.

Corollary 1: If $k \geq n$, then the asymptotic behavior of the expected complementary cumulative in-degree distribution satisfies

$$F^c_\infty(k) = \begin{cases} \frac{\Gamma\left(k+\frac{mrp_r}{cp_c}\right)\Gamma\left(n+\frac{m(1+rp_r)}{cp_c}\right)}{\Gamma\left(n+\frac{mrp_r}{cp_c}\right)\Gamma\left(k+\frac{m(1+rp_r)}{cp_c}\right)} & \text{if } cp_c \neq 0 \\ 1 - \left(\frac{rp_r}{1+rp_r}\right)^{k-n} & \text{if } cp_c = 0 \end{cases}$$

Proof: Let $P_\infty(k) = \lim_{t\to\infty} P_t(k)$. Using Theorem 1, we know that

$$P_\infty(k) = \begin{cases} \frac{rp_r + \frac{(k-1)cp_c}{m}}{1+rp_r+\frac{kcp_c}{m}} P_\infty(k-1) & \text{if } k > n \\ \frac{1}{1+rp_r+\frac{ncp_c}{m+n}} & \text{if } k = n \end{cases}$$

Note that $P_\infty(k)$ can be expressed in terms of a falling factorial as

$$P_\infty(k) = \frac{1}{rp_r + \frac{kcp_c}{m}} \prod_{j=n}^{k} \frac{rp_r + \frac{jcp_c}{m}}{1 + rp_r + \frac{jcp_c}{m}}$$

Using the gamma function representation, the above expression can be written as

$$P_\infty(k) = \begin{cases} \frac{m\Gamma\left(k+\frac{mrp_r}{cp_c}\right)\Gamma\left(n+\frac{m(1+rp_r)}{cp_c}\right)}{cp_c\Gamma\left(n+\frac{mrp_r}{cp_c}\right)\Gamma\left(k+\frac{m(1+rp_r)}{cp_c}\right)} & \text{if } cp_c \neq 0 \\ \frac{1}{1+rp_r}\left(\frac{rp_r}{1+rp_r}\right)^{k-n} & \text{if } cp_c = 0 \end{cases} \qquad (7)$$

Because $F^c_\infty(k) = P[K_\infty \geq k] = 1 - \sum_{j=n}^{k-1} P_\infty(j)$, using eq. (7) we get the desired result. ■

The next section characterizes the expected average in-degree of the network, which we require to prove the stability properties of the in-degree distribution.

III. Stability of the in-degree distribution

Note that the expected value of the average in- and the out-degree are equal. Let d_0 represent the sum of all in-degree of the nodes of the initial network. Moreover, let $\bar{K}_t$ be a random variable that characterizes the average in-degree of $\mathcal{G}_t$. Note that

$$\bar{K}_t = \frac{d_0 + \sum_{i=1}^t M_i + tn}{n_0 + t} \qquad (8)$$

Using the Law of Large Numbers, we know that

$$\begin{aligned} \lim_{t\to\infty} \bar{K}_t &= \lim_{t\to\infty} \frac{d_0 + \sum_{i=1}^t M_i + tn}{n_0 + t} \\ &= \lim_{t\to\infty} \frac{\sum_{i=1}^t M_i + tn}{t} \\ &= m \end{aligned}$$

Note that

$$\begin{aligned} \mathrm{E}[\bar{K}_t] &= \mathrm{E}\left[\frac{d_0 + \sum_{i=1}^t M_i + tn}{n_0 + t}\right] \\ &= \frac{d_0 + tm}{n_0 + t} \qquad (9) \end{aligned}$$

Based on eq. (9), $\mathrm{E}[\bar{K}_\infty] = \lim_{t\to\infty} \mathrm{E}[\bar{K}_t] = m$ and

$$\mathrm{E}[\bar{K}_{t+1}] = \frac{(n_0 + t)\mathrm{E}[\bar{K}_t] + \mathrm{E}[M_t] + n}{n_0 + t + 1} \qquad (10)$$

Remark 1: The monotonicity of the expected average in-degree depends on the average in-degree of the initial network. Note that $\mathrm{E}[\bar{K}_t]$ is strictly increasing if $\bar{K}_0 < \mathrm{E}[\bar{K}_\infty]$; strictly decreasing if $\bar{K}_0 > \mathrm{E}[\bar{K}_\infty]$; equal to $\mathrm{E}[\bar{K}_\infty]$ if $\bar{K}_0 = \mathrm{E}[\bar{K}_\infty]$.

A. Stability of the complementary cumulative in-degree distribution

Define the distribution of the network as an infinite dimensional vector $x_t = (x_t(0), x_t(1), \ldots)$, where $x_t(k)$ represents the probability that the expected in-degree of a node, selected uniformly at random at time t, is greater than or equal to k. That is, $x_t(k) = P(\mathrm{E}[K_t] \geq k)$. Let $x^e = (x^e(0), x^e(1), \ldots)$ denote the limit distribution with $x^e(k) = F^c_\infty(k)$. Note that

$$\begin{aligned}\sum_{k=0}^{\infty} x_t(k) &= \sum_{k=0}^{\infty}(k+1)P(\mathrm{E}[K_t]=k) \\ &= \sum_{k=0}^{\infty} kP(\mathrm{E}[K_t]=k) + \sum_{k=0}^{\infty} P(\mathrm{E}[K_t]=k) \\ &= \mathrm{E}[\bar{K}_t] + 1 \end{aligned} \tag{11}$$

Using eq. (9), note also that

$$\lim_{t\to\infty} \sum_{k=0}^{\infty} x_t(k) = \sum_{k=0}^{\infty} x^e(k) = \mathrm{E}[\bar{K}_\infty] + 1$$

Define $\mathcal{X}$ as the set of all bounded sequences in $[0,1]$ such that the only sequence that satisfies $\sum_{k=0}^{\infty} x(k) = \mathrm{E}[\bar{K}_\infty]+1$ is F^c_∞, i.e.,

$$\mathcal{X} = \left\{ x \in [0,1]^\infty : \sum_{k=0}^{\infty} x(k) = \mathrm{E}[\bar{K}_\infty] + 1 \Rightarrow x = F^c_\infty \right\}$$

Let $\mathcal{X}_0 \subseteq \mathcal{X}$ be the set of all possible initial distributions. For $x_0 \in \mathcal{X}_0$, there exists a network $\mathcal{G}_0$ with complementary cumulative in-degree distribution x_0. Define the set

$$\mathcal{X}_C = \left\{ x \in \mathcal{X} : \sum_{k=0}^{\infty} x(k) = \mathrm{E}[\bar{K}_\infty] + 1 \right\} \tag{12}$$

Note that $\mathcal{X}_C = \{x^e\}$. Moreover, because $\lim_{t\to\infty} x_t(k) = F^c_\infty(k)$, $\mathcal{X}_C$ corresponds to a positive limit set of the model. Using Lemma 3.1 in [14], it can be shown that $\mathcal{X}_C$ is an invariant set.

To guarantee that $\mathcal{X}_C$ is asymptotically stable, we introduce the following distance function on $\mathcal{X}$.

Lemma 2: Consider the function $\rho : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+_0$,

$$\rho(x,y) = \left| \sum_{k=0}^{\infty} (x(k) - y(k)) \right|$$

and define an equivalence relation on $\mathcal{X}$ as x being related to y if $\rho(x,y) = 0$. Let $[\![x]\!]$ denote the equivalence class of x and $\mathcal{X}^* = \{[\![x]\!] : x \in \mathcal{X}\}$ the set of all equivalence classes. If $\rho^* : \mathcal{X}^* \times \mathcal{X}^* \to \mathbb{R}^+_0$ is defined as $\rho^*([\![x]\!], [\![y]\!]) = \rho(x,y)$, then $(\rho^*, \mathcal{X}^*)$ is a metric space.

Proof: Let $w, x, y, z \in \mathcal{X}$. First, we show that ρ is a pseudometric. In particular, note that $\rho(x,y) \geq 0$ and $\rho(x,y) = \rho(y,x)$. To verify that ρ satisfies the triangle inequality, note that

$$\begin{aligned}\rho(x,y) &= \left| \sum_{k=0}^{\infty} (x(k) - y(k)) \right| \\ &= \left| \sum_{k=0}^{\infty} (x(k) - z(k)) + \sum_{k=0}^{\infty} (z(k) - y(k)) \right| \\ &\leq \rho(x,z) + \rho(z,y)\end{aligned}$$

In general, for $x, y \in \mathcal{X}$, note that $x \neq y$ does not imply that $\rho(x,y) \neq 0$ (i.e., ρ is a pseudometric). Second, we show that for the equivalence relation over $\mathcal{X}$, ρ^* is well-defined, that is, if $([\![x]\!], [\![y]\!]) = ([\![z]\!], [\![w]\!])$ then $\rho^*([\![x]\!], [\![y]\!]) = \rho^*([\![z]\!], [\![w]\!])$. In particular, if $([\![x]\!], [\![y]\!]) = ([\![z]\!], [\![w]\!])$, then $[\![x]\!] = [\![z]\!]$ and $[\![y]\!] = [\![w]\!]$. Because ρ satisfies the triangle inequality, note that $\rho(x,y) \leq \rho(z,w)$ and $\rho(z,w) \leq \rho(x,y)$, that is, $\rho(x,y) = \rho(z,w)$, which implies that $\rho^*([\![x]\!], [\![y]\!]) = \rho^*([\![z]\!], [\![w]\!])$.

Finally, we present sufficient conditions for $(\rho^*, \mathcal{X}^*)$ to be a metric space. Let $[\![x]\!], [\![y]\!], [\![z]\!] \in \mathcal{X}^*$. In particular, since ρ is a pseudometric, note that ρ^* satisfies:

1) For $[\![x]\!] \neq [\![y]\!]$, we know that

$$\rho^*([\![x]\!], [\![y]\!]) = \rho(x,y) = \left| \sum_{k=0}^{\infty} (x(k) - y(k)) \right| > 0$$

2) For $x \in [\![x]\!]$ and $y \in [\![y]\!]$, note that $\rho^*([\![x]\!], [\![y]\!]) = 0$ if and only if $\rho(x,y) = 0$, that is, if and only if $|\sum_{k=0}^{\infty}(x(k) - y(k))| = 0$. This implies that $y \in [\![x]\!]$ and $x \in [\![y]\!]$. Therefore, $\rho^*([\![x]\!], [\![y]\!]) = 0$ if and only if $[\![x]\!] = [\![y]\!]$.
3) For $x, y \in \mathcal{X}$, we know that $\rho^*([\![x]\!], [\![y]\!]) = \rho^*([\![y]\!], [\![x]\!])$ because $\rho(x,y) = \rho(y,x)$.
4) For $x, y, z \in \mathcal{X}$

$$\begin{aligned}\rho^*([\![x]\!], [\![y]\!]) &= \rho(x,y) \\ &\leq \rho(x,z) + \rho(z,y) \\ &= \rho^*([\![x]\!], [\![z]\!]) + \rho^*([\![z]\!], [\![y]\!])\end{aligned}$$

■

We use Lemma 2 to prove the stability of $\mathcal{X}_C$.

Theorem 2: The invariant set $\mathcal{X}_C$ is asymptotically stable with basin of attraction $\mathcal{X}_0$.

Proof: Let $\mathcal{V} : \mathcal{X}^* \to \mathbb{R}^+_0$, defined by $\mathcal{V}([\![x]\!]) = \rho^*([\![x]\!], [\![x^e]\!])$, be a Lyapunov candidate function. Because $x^e \in \mathcal{X}$ is the only distribution that satisfies $\sum_{k=0}^{\infty} x(k) = \mathrm{E}[\bar{K}_\infty] + 1$, we have that $[\![x^e]\!] = \{x^e\}$, which implies that $\mathcal{X}_C = [\![x^e]\!]$. Note that $\mathcal{V}(\mathcal{X}_C) = 0$. The following four conditions are sufficient to guarantee the asymptotic stability of $\mathcal{X}_C$ [15].

Existence of a lower bound: From the definition of $\mathcal{V}$, we have that for all sufficiently small $\varepsilon_1 > 0$, there exists a $\delta_1 = \varepsilon_1$ such that for all $[\![x]\!] \in \mathcal{X}^*$, if $\rho^*([\![x]\!], \mathcal{X}_C) > \varepsilon_1$, then $\mathcal{V}([\![x]\!]) > \delta_1$.

Existence of an upper bound: Note that for all sufficiently small $\varepsilon_2 > 0$, there exists a $\delta_2 = \varepsilon_2$ such that for all $[\![x]\!] \in$

$\mathcal{X}^*$, if $\rho^*([\![x]\!], \mathcal{X}_C) < \delta_2$, then $\mathcal{V}([\![x]\!]) \leq \varepsilon_2$.

Non-increasing trajectories along all possible motions: Consider the following two cases based on the value of $\bar{K}_0$. Using Remark 1, if $\bar{K}_0 < \mathrm{E}[\bar{K}_\infty]$, then

$$\begin{aligned}\mathcal{V}([\![x_t]\!]) &= \left|\sum_{k=0}^{\infty} x_t(k) - \sum_{k=0}^{\infty} x^e(k)\right| \\ &= \mathrm{E}[\bar{K}_\infty] + 1 - \sum_{k=0}^{\infty} x_t(k)\end{aligned}$$

Because $\mathrm{E}[\bar{K}_t]$ is a strictly increasing function, using eq. (11), we have that

$$\mathcal{V}([\![x_t]\!]) - \mathcal{V}([\![x_{t+1}]\!]) = \mathrm{E}[\bar{K}_{t+1}] - \mathrm{E}[\bar{K}_t] > 0$$

Similarly, if $\bar{K}_0 > \mathrm{E}[\bar{K}_\infty]$ then

$$\mathcal{V}([\![x_t]\!]) - \mathcal{V}([\![x_{t+1}]\!]) = \mathrm{E}[\bar{K}_t] - \mathrm{E}[\bar{K}_{t+1}] > 0$$

Convergence: Because $x_t(k) = P[\mathrm{E}[K_t] \geq k] = F_t^c(k)$, we know that $\lim_{t\to\infty} x_t(k) = F_\infty^c(k) = x^e(k)$, which implies that $\mathcal{V}([\![x_t]\!]) \to 0$ as $t \to \infty$.

∎

B. Stability of the average in-degree

We now characterize an invariant set that captures the behavior of the in-degree distribution. Let $y_t = \mathrm{E}[\bar{K}_t]$ denote the expected value of the average in-degree and $y^e = \lim_{t\to\infty} y_t = \mathrm{E}[\bar{K}_\infty]$. Let $d(\cdot,\cdot)$ denote the Euclidean distance and

$$\mathcal{Y}_C = \{y \in \mathbb{R}_0^+ : d(y, y^e) = 0\}$$

It is clear that $\{y^e\} = \mathcal{Y}_C$. To show that $\mathcal{Y}_C$ is an invariant set, suppose that $y_t = y^e$ at $t \geq 0$. Note that $\mathrm{E}[\bar{K}_t] = \mathrm{E}[\bar{K}_\infty]$. Based on eq. (10), we know that $\mathrm{E}[\bar{K}_{t+1}] = \mathrm{E}[\bar{K}_t]$, that is, $y_{t+1} = y^e$, which implies that $\mathcal{Y}_C$ is an invariant set.

The following theorem proves the stability of $\mathcal{Y}_C$.

Theorem 3: The set $\mathcal{Y}_C$ is asymptotically stable with basin of attraction $\mathbb{R}^+$.

Proof: Consider the function

$$\bar{\rho}(y, \mathcal{Y}_C) = \inf\{d(y,u) : u \in \mathcal{Y}_C\}$$

and let $\mathcal{W}(y) = \bar{\rho}(y, \mathcal{Y}_C)$ be a Lyapunov function candidate. Note that $\bar{\rho}(y, \mathcal{Y}_C) = d(y, y^e)$. Furthermore, we know that $\mathcal{W}(y) \geq 0$ for all $y \in \mathcal{Y}$, and $\mathcal{W}(y) = 0$ if and only if $y \in \mathcal{Y}_C$. The following conditions guarantee the asymptotic stability of $\mathcal{Y}_C$.

Existence of a lower bound: Note that for all sufficiently small $\varepsilon_1 > 0$, there exists a $\delta_1 = \varepsilon_1$ such that for any $y \in \mathbb{R}_0^+$, if $\bar{\rho}(y, \mathcal{Y}_C) > \varepsilon_1$, then $\mathcal{W}(y) > \delta_1$.

Existence of an upper bound: Note that for all sufficiently small $\varepsilon_2 > 0$, there exists a $\delta_2 = \varepsilon_2$ such that for any $y \in \mathbb{R}_0^+$, if $\bar{\rho}(y, \mathcal{Y}_C) < \delta_2$, then $\mathcal{W}(y) \leq \varepsilon_2$.

Non-increasing trajectories along all possible motions: Because $\mathcal{Y}_C$ is an invariant set, if $y_t \in \mathcal{Y}_C$, then $y_{t'} \in \mathcal{Y}_C$ for all $t' > t$, which implies that $\mathcal{W}(y_t) = \mathcal{W}(y_{t'}) = 0$. Now, suppose that $y_t \notin \mathcal{Y}_C$. If $y_{t+1} \in \mathcal{Y}_C$, then $\mathcal{W}(y_t) > \mathcal{W}(y_{t+1}) = 0$. If $y_{t+1} \notin \mathcal{Y}_C$, then

$$\mathcal{W}(y_t) - \mathcal{W}(y_{t+1}) = \frac{n_0|\bar{K}_0 - \mathrm{E}[\bar{K}_\infty]|}{(n_0+t)(n_0+t+1)} > 0$$

That is, $\mathcal{W}(y_t) > \mathcal{W}(y_{t+1})$, which implies that $\mathcal{W}$ is non-increasing along all possible motions of the model.

Convergence: Because $y_t = \mathrm{E}[\bar{K}_t]$, $\lim_{t\to\infty} y_t = y^e$, which implies that $\mathcal{W}(y_t) \to 0$ as $t \to \infty$.

∎

IV. Simulations

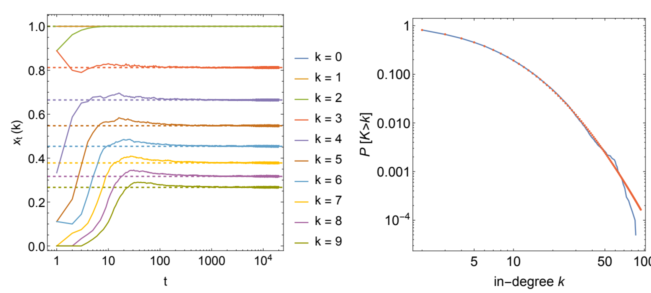
Let $r = 5$, $c = 3$, $p_r = 0.8$, $p_c = 0.4$ and $n = 2$. Let $\mathcal{G}_0 = (V_0, E_0)$ be an initial network with $|V_0| = 9$ and in-degrees $(3,3,3,3,1,3,6,4,4)$. The initial state is given by

$$x_0 = \{1, 1, 0.88, 0.88, 0.33, 0.11, 0.11, 0, 0, \ldots\}$$

According to eq. (12), the invariant set $\mathcal{X}_C$ is given by

$$\mathcal{X}_C = \{x \in \mathcal{X} : \textstyle\sum_{k=0}^{\infty} x(k) = 8.2\}$$

Figure 1 illustrates the evolution of the states of the simulated network for the in-degree together with the theoretical values of $x^e(k)$ for $0 \leq k \leq 9$. Note that the simulated distributions approach the theoretical limits (based on Corollary 1). Simulations correspond to an average of 100 runs of the model.



(a) (b)

Fig. 1. (a) Evolution of $x_t(k) = P[\mathrm{E}[K_t] \geq k]$ for $0 \leq k_0 \leq 9$; and (b) Complementary cumulative in-degree distribution. Solid line represents the average of the ccdf of 100 runs of the model and the dashed represents the predictions for $r = 5$, $c = 3$, $p_r = 0.8$, $p_n = 0.4$ and $n = 2$.

V. Conclusions

Based on the discrete version of the Jackson-Rogers model, our work uses a discrete-time approach to characterize the evolution of the probability function that a node has particular in-degree k at time t. Moreover, we describe the asymptotic behavior for the cumulative in-degree distribution, and show that the distribution is asymptotically stable. We also show that the average in-degree is asymptotically stable. Characterizing the stability properties of other centrality measures, for example, the eigenvector centrality, remains a future research direction.

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