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Ideal Free Distributions in Human Decision–Making

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***Abstract*— Integrating human factors into the design of large-scale distributed applications requires capturing broad patterns of decision-making over time. The proposed theoretical framework introduces a dynamic model that resembles empirical dispersal patterns between the quality of an option and the number of individuals choosing that option. We use the notion of the Ideal Free Distribution (IFD) to estimate the resulting population-dependent equilibrium point and reduce uncertainty about how groups of individuals choose between available options. Our contribution is twofold. First, we identify conditions that lead to the IFD under constrained choice. Second, we illustrate how biases in decision-making can lead to systemic deviations from the IFD.**

I. INTRODUCTION

The well-known concept in behavioral ecology of the Ideal Free Distribution (IFD) characterizes how foragers allocate themselves to different resources under two main assumptions [1]. The word *ideal* suggests that all foragers have complete knowledge and equal abilities to compete for a finite number of resources across different sites. The word *free* suggests that foragers can move between any two sites without incurring any cost. In a closed environment (i.e., with a constant population), foragers reach an equilibrium distribution where all foragers have an equal chance to succeed. The probability of success is determined by $\frac{w}{m}$ where w and m are the number of resources and foragers at a site. According to the IFD the allocation of foragers at any two sites i and j ultimately satisfies

$$\frac{m_i}{m_j} = \frac{w_i}{w_j} \tag{1}$$

Because the dispersal pattern across sites is not random, empirical distributions resembling the IFD suggest the existence of decision-making mechanisms that are influence by social and environmental determinants of individual success. Resting on the ideal-free assumptions, the main insight behind the IFD is that decision-making not only depends on the number of resources available, but also on the number of individuals allocating their efforts to a site. The IFD can be viewed as an equilibrium point that is Pareto optimal in the sense that no individual, as a player in a social interaction game, can benefit from changing its strategy unilaterally without making herself and other players worse off (indeed it represents an optimal mean benefit for all players). Population-dependent equilibria and systemic deviations from the IFD are often described by

$$\log \frac{m_i}{m_j} = a \log \frac{w_i}{w_j} + \log b_{ij} \tag{2}$$

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where $a > 0$ is a measure of sensibility between a group of individuals and a set of resources, and $b_{ij} > 0$ captures an underlying preference for one site over another [2]. Equation (2) (called the *generalized habitat matching rule*) represents how individuals distribute themselves in direct relation to the number of resources.

The matching rule captures a variety of empirical distributions that result of the dynamics of resource use. In [3] the authors identify regularities that resemble IFD-like patterns in decisions among humans. They carry out an experiment in which participants are invited to choose between two groups. A fixed but different number of participants are randomly selected from each group and awarded one point. After ten rounds, the participant with the most points wins a reward. The results of this experiment show that the distribution of members (the so-called group choice) is proportional to the expected number of participants selected from each group. In other words, group choice results from decision-making mechanisms that relate the choice ratio between groups $\frac{m_i}{m_j}$ to the ratio of points associated to the groups $\frac{w_i}{w_j}$ according to eq. (2).

To understand systemic deviations from the IFD, the authors of [4] investigate the sensitivity of eq. (2) with respect to the parameter a. When $a = 1$, the ratio of resources matches the ratio of members in a group competing for shared resources (known as strict matching). When $a < 1$, the model indicates that the ratio of members in a group is less than the ratio of resources associated to that group, a result called *undermatching*. Undermatching represents a distribution of individuals that is less extreme than the distribution of resources, in the sense that there are fewer individuals at the sites with the most resources. Empirical measures suggest that human distributions are indeed less sensitive to resource allocation (with sensitivity parameters in the range $0.5 \leq a \leq 0.9$ [3]–[10]). In other words, individuals tend to perceive sites with the most resources as less attractive than predicted by strict matching ($a = 1$). Modeling dynamics that yield IFD-like patterns are a valuable tool to help explain undermatching as the natural outcome of decision-making processes in which competitors underuse an available set of resources.

In general, both human factors (perception biases and decision heuristics) and environmental constraints seem to account for systemic deviations of group choice from the ideal matching rule. Ideal free distributions in human decision-making are important in the process of (i) identifying what particular determinants affect resource competition; and (ii) integrating human factors into the design of large-scale applications on interacting platforms (e.g., network-based applications that base their services on broad empirical patterns of collective decision-making). It is the nature of such research addressing issues of human interaction to understand the particular social phenomenon and to engineer systems that focus and take advantage of it.

The remainder of this paper is structured as follows. Section II proposes a model that captures the collective dynamics of decision-making when individuals compete for shared resources. The model offers an analytical framework that captures the phenomenon of undermatching (rather than the perfect IFD) as the outcome of the interaction of rational agents (i.e., individuals trying to maximize a utility function associated to a set of options). Theorem 1 in Section III presents conditions that guarantee that an invariant set representing eq. (2) is asymptotically stable. Simulation results in Section IV focus on the effects of decision-making on the sensitivity parameter. Finally, Section V draws some conclusions and future research directions.

II. THE MODEL

Consider an undirected network $\mathcal{G} = (\mathcal{N}, \mathcal{A})$, where $\mathcal{N} = \{1, \ldots, n+1\}$ represents the set of $n+1$ nodes (available options) and $\mathcal{A}$ the set of edges (i.e., the possibility of switching between options). The network represents the constraints in the choices of $q > 1$ individuals, each allowed to select an option (a node) at no cost but only according to the set $\mathcal{A}$. If $i \in \mathcal{N}$, we say a node $j \in \mathcal{N}$ is adjacent to i if $\{i, j\} \in \mathcal{A}$. The neighborhood of node i is defined as $\mathcal{N}_i := \{j \in \mathcal{N} : \{i, j\} \in \mathcal{A}\}$. Note that modeling the dynamics of decision-making on a network relaxes to some extent the ideal and free conditions of the original assumptions (e.g., some individuals may not be aware or able to choose particular options depending on their current choice).

It is common to consider a large value of $q = m_1 + \cdots + m_{n+1}$ so that the distribution of choices across options is a function of the fraction of individuals selecting the particular options. Define $\mathbf{x} = (x_1, \ldots, x_n)$ to be the state of the system where $x_i = \frac{m_i}{q}$, $x_i \in \mathcal{R}, \mathcal{R} = (0, 1]$, represents the proportion of individuals selecting option i. Let Δ be the simplex of all $n+1$ tuples on $\mathcal{R}^{n+1}$, that is

$$\Delta := \left\{ (x_1, \ldots, x_{n+1}) \in \mathcal{R}^{n+1} : \sum_{i \in \mathcal{N}} x_i = 1 \right\} \tag{3}$$

Because it only requires n states to represent the dynamics on $\mathcal{G}$, we define the set $\mathcal{X}$ as the projection of the simplex Δ on the n-dimensional space $x_1 \ldots x_n$

$$\mathcal{X} := \{(x_1, \ldots, x_n) \in \mathcal{R}^n : (x_1, \ldots, x_{n+1}) \in \Delta\} \tag{4}$$

Consider the following assumptions on the network and the utility function $u_i(x_i)$ associated to each node $i \in \mathcal{N}$.

(A1) The network is connected, without self-loops or parallel edges.

(A2) The utility function u_i is increasing, continuously differentiable, strictly concave on $\mathcal{R}$, and the derivative with respect to x_i (its marginal utility function), $s_i := u_i'$, is Lipschitz continuous on $\mathcal{R}$.

Assumption (A1) places minimum constraints on the way individuals may switch between options. Assumption (A2) implies that for each $i \in \mathcal{N}$ there exists a positive constant K_i such that

$$\frac{|s_i(x) - s_i(y)|}{|x - y|} \leq K_i \tag{5}$$

for all $x, y \in \mathcal{R}$, $x \neq y$. Moreover, because s_i is strictly decreasing in $\mathcal{R}$ then

$$\frac{s_i(x) - s_i(y)}{x - y} < 0 \tag{6}$$

The proposed decision-making mechanism rests on the following family of marginal utility functions

$$s_i(x_i) := \frac{w_i^a}{x_i} \tag{7}$$

where $w_i > 0$ is the number of resources available at node i and a is the sensitivity parameter from eq. (2). Because different options represent different possibilities of success (e.g., different number of resources are available for different options), note that in general $w_i \neq w_j$ for some $i, j \in \mathcal{N}$. According to eq. (7), the utility function of node i (with constant of integration equal to zero) is given by

$$u_i(x_i) := \int s_i(x_i) dx_i = w_i^a \ln(x_i)$$

The utility function for the entire group of decision-makers is defined as

$$\begin{array}{rccc} u: & \mathcal{R}^{n+1} & \to & \mathbb{R}_0^+ \\ & (x_1, \ldots, x_{n+1}) & \mapsto & \sum\limits_{i \in \mathcal{N}} u_i(x_i) \end{array}$$

where $\mathbb{R}_0^+$ represents the set of non-negative real numbers.

To show that the utility functions satisfy Assumption (A2), note, first, that because the marginal functions are well defined and continuous on $\mathcal{R}$, u_i is continuously differentiable on $\mathcal{R}$. Second, because u_i is continuously differentiable on $\mathcal{R}$ and $s_i(x_i) > 0$ for all $x_i \in \mathcal{R}$, then u_i is strictly increasing on $\mathcal{R}$. And third, the second derivative of u_i

$$u_i'' = -\frac{w_i^a}{x_i^2}$$

is always negative on $\mathcal{R}$. Thus, u_i is strictly concave on $\mathcal{R}$.

With respect to the marginal utility of u_i, note that for $x, y \in \mathcal{R}$

$$\left| \frac{w_i^a}{x} - \frac{w_i^a}{y} \right| = |w_i^a| \left| \frac{y - x}{xy} \right|$$

And because $x, y \geq \frac{1}{q}$, we know that

$$|s_i(x) - s_i(y)| \leq q^2 w_i^a |x - y|$$

Thus, s_i is Lipschitz continuous on $\mathcal{R}$ and satisfies eq. (5) with $K_i = q^2 w_i^a$. Finally, using the definition of s_i, it can be shown that there exists a positive constant L_i such that

$$\frac{s_i(x) - s_i(y)}{x - y} = -\frac{w_i^a}{xy} \leq -\frac{w_i^a}{q^2} = -L_i$$

for all $x, y \in \mathcal{R}$, $x \neq y$. To satisfy eq. (6) let $L := \min\{L_i : i \in \mathcal{N}\}$ and $K := \max\{K_i : i \in \mathcal{N}\}$. In particular

$$-K \leq \frac{s_i(x) - s_i(y)}{x - y} \leq -L \tag{8}$$

Assumption (A2) guarantees that there exists a unique state $\mathbf{x}^* = (x_1^*, \ldots, x_n^*)$ such that $u(x_1^*, \ldots, x_{n+1}^*)$ is maximized if and only if $s_i(x_i^*) = s_j(x_j^*)$ for all $i, j \in \mathcal{N}$ [11]. Let the set

$$\mathcal{X}^* = \{\mathbf{x}^* \in \mathcal{X} : s_i(x_i^*) = s_j(x_j^*), \forall i, j \in \mathcal{N}\} \tag{9}$$

For convenience we denote $s^* = s_i(x_i^*)$ for $i \in \mathcal{N}$. The dynamics of decision-making are governed by a discrete event system represented by $\mathcal{S} = (\mathcal{X}, \mathcal{G}, \mathcal{E}, g, f_e)$, where $\mathcal{E}$ denotes the set of all possible events that drive the dynamics of the choices by the decision-makers. We denote an event at time index k as $e(k)$. Transitions between states depend on the occurrence of events denoted by the activation function g. For $\mathbf{x}(k) \in \mathcal{X}$, we say that an event $e(k)$ is active at k, if $e(k) \in g(\mathbf{x}(k))$. Note that at time index k there is just one state, but there could be many active events. If an active event $e(k)$ occurs at k, the transition function f_e generates the state $\mathbf{x}(k+1)$ defined by $\mathbf{x}(k+1) := f_{e(k)}(\mathbf{x}(k))$. If there is a deadlock at k, the only active event is the null event e^0, where $f_{e^0}(\mathbf{x}(k)) = \mathbf{x}(k)$.

To define the set of events $\mathcal{E}$, let

$$\mathcal{M}_i := \{j \in \mathcal{N}_i : s_i(x_i) > s_j(x_j)\}$$

be the set of nodes $j \in \mathcal{N}_i$ such that the marginal utility $s_i(x_i)$ is greater than $s_j(x_j)$. The set $\mathcal{M}_i$ represents all nodes for which node i is a feasible and attractive alternative choice.

With e_{ji}, $j \in \mathcal{M}_i$, we represent changing option j for option i. Let $\mathcal{E}_\alpha = \{e_{ji}\}$ be the set of all possible reallocation of choices. Then, the set of events is given by the powerset of $\mathcal{E}_\alpha$ (without the empty set), $\mathcal{E} = P(\mathcal{E}_\alpha) \setminus \{\emptyset\}$, and an event $e(k) \in \mathcal{E}$ is a set where each element represents the reallocation of choices.

Changing option j for option i for a fraction of γ_{ji} decision-makers is denoted

$$\gamma_{ji} = \frac{1}{2}\left(x_j - \left(\frac{w_j}{w_i}\right)^a x_i\right) \tag{10}$$

where $a > 0$ can be viewed as the group's ability to differentiate between environmental factors w_j and w_i. As $a \to 0$ individuals tend to neglect the number of resources associated to an option and focus solely on the number of individuals choosing the same option. If an active event $e(k)$ occurs with $e_{ji} \in e(k)$ and $\gamma_{ji} > 0$, then we consider $\gamma_{ji'} = 0$ for all $e_{ji'} \in e(k)$.

The proportion of decision-makers choosing option i at time index $k+1$ is

$$x_i(k+1) = x_i(k) - \sum_{\{j:e_{ij} \in e(k)\}} \gamma_{ij} + \sum_{\{j:e_{ji} \in e(k)\}} \gamma_{ji} \tag{11}$$

III. STABILITY PROPERTIES OF THE MODEL

The following theorem establishes the stability properties of $\mathcal{X}^*$. It shows that starting from any initial state the distribution of choices across $\mathcal{G}$ leads to the IFD.

Theorem 1: The invariant set $\mathcal{X}^*$ defined in eq. (9) has a region of asymptotic stability equal to $\mathcal{X}$.

To prove the stability properties of $\mathcal{X}^*$, consider the Lyapunov candidate function

$$\mathcal{V}(x_1, \ldots, x_n) := \max\{s_1(x_1), \ldots, s_{n+1}(x_{n+1})\} - s^* \tag{12}$$

and define $\rho(\mathbf{x}, \mathcal{X}^*) := \inf\{\rho(\mathbf{x}, \mathbf{x}^*) : \mathbf{x}^* \in \mathcal{X}^*\}$ as the distance from $\mathbf{x}$ to $\mathcal{X}^*$[1]. Let the r-neighborhood of $\mathcal{X}^*$ be $B(\mathcal{X}^*; r) := \{\mathbf{x} \in \mathcal{X} : 0 < \rho(\mathbf{x}, \mathcal{X}^*) < r\}$. Suppose that $\mathbf{x} \in B(\mathcal{X}^*; r)$, $\rho(\mathbf{x}, \mathcal{X}^*) = |x_i - x_i^*|$ (i.e., node i represents the node with the largest distance to $\mathcal{X}^*$) and $\max\{s_1(x_1), \ldots, s_{n+1}(x_{n+1})\} = s_j(x_j)$ (i.e., node j represents a node with the highest marginal utility value). The following four conditions are sufficient to guarantee the stability properties of $\mathcal{X}^*$.

1) For all sufficiently small $\varepsilon_1, 0 < \varepsilon_1 < r$, there exists a $\delta_1 > 0$ such that for all $\mathbf{x} \in \mathcal{X}$

$$(\varepsilon_1 < \rho(\mathbf{x}, \mathcal{X}^*) < r) \Rightarrow \mathcal{V}(\mathbf{x}) > \delta_1$$

2) For all sufficiently small $\varepsilon_2 > 0$, there exists a $\delta_2 > 0$ such that for all $\mathbf{x} \in \mathcal{X}$

$$(\rho(\mathbf{x}, \mathcal{X}^*) < r \wedge \rho(\mathbf{x}, \mathcal{X}^*) < \delta_2) \Rightarrow \mathcal{V}(\mathbf{x}) > \varepsilon_2$$

3) $\mathcal{V}$ is a non-increasing function.
4) $\mathcal{V}(\mathbf{x}(k)) \to 0$ as $k \to \infty$.

Proof:

1) To show that for all sufficiently small $\varepsilon_1, 0 < \varepsilon_1 < r$, there exits a $\delta_1 > 0$, such that for all $\mathbf{x} \in \mathcal{X}$, if $\varepsilon_1 < \rho(\mathbf{x}, \mathcal{X}^*) < r$ then $\mathcal{V}(\mathbf{x}) > \delta_1$ consider the following two cases. If $i = j$, we have $x_i - x_i^* < 0$, and according to eq. (8) there exist two constants $K, L > 0$ such that

$$-K(x_i - x_i^*) \geq \mathcal{V}(\mathbf{x}) \geq -L(x_i - x_i^*) = L(x_i^* - x_i)$$

Thus $\mathcal{V}(\mathbf{x}) \geq L(x_i^* - x_i) = L\,|x_i - x_i^*| > L\varepsilon_1 = \delta_1$.
If $i \neq j$, then $|x_i - x_i^*| \geq |x_j - x_j^*|$ and $x_j - x_j^* < 0$. Using again eq. (8) we have

$$-K(x_j - x_j^*) \geq \mathcal{V}(\mathbf{x}) \geq -L(x_j - x_j^*)$$

Because $s_j(x_j) \geq s_i(x_i)$

$$s_j(x_j) - s_j(x_j^*) \geq s_i(x_i) - s_i(x_i^*) \tag{13}$$

Suppose $x_i - x_i^* < 0$. Then $s_i(x_i) - s_i(x_i^*) \geq L\,|x_i - x_i^*|$.
Using eq. (13) we have

$$s_j(x_j) - s_j(x_j^*) = \mathcal{V}(\mathbf{x}) \geq L\,|x_i - x_i^*| > L\varepsilon_1 = \delta_1$$

[1] $\rho(\mathbf{x}, \mathbf{y}) := \max\{|x_i - y_i| : i = 1, \ldots, n\}$ is the metric on $\mathcal{X}$, where $\mathbf{y} = (y_1, \ldots, y_n) \in \mathcal{X}$. Because $\mathcal{X}^*$ is a singleton set, the distance from $\mathbf{x}$ to $\mathcal{X}^*$ is given by $\rho(\mathbf{x}, \mathcal{X}^*) := \rho(\mathbf{x}, \mathbf{x}^*)$.

Now suppose $x_i - x_i^* > 0$. Then $s_i(x_i) < s^*$ which means some individuals must find it attractive to change option i for an alternative option. Let

$$\mathcal{D}_1 := \{\ell \in \mathcal{N} : s_\ell(x_\ell) > s^*\}$$
$$\mathcal{D}_2 := \{\ell \in \mathcal{N} : s_\ell(x_\ell) < s^*\}$$

be the sets of nodes with marginal utilities above and below of the equilibrium value, respectively. It is clear that

$$\sum_{\ell \in \mathcal{D}_2} (x_\ell - x_\ell^*) = \sum_{\ell \in \mathcal{D}_1} (x_\ell^* - x_\ell) \tag{14}$$

Moreover, using eq. (8), for each $\ell \in \mathcal{D}_1$ we have

$$L(x_\ell^* - x_\ell) \le s_\ell(x_\ell) - s_\ell(x_\ell^*) \le K(x_\ell^* - x_\ell) \tag{15}$$

And combining eq. (14) and eq. (15) we have

$$\begin{aligned} x_i - x_i^* &\le \sum_{\ell \in \mathcal{D}_2} (x_\ell - x_\ell^*) \\ &= \sum_{\ell \in \mathcal{D}_1} (x_\ell^* - x_\ell) \\ &\le \sum_{\ell \in \mathcal{D}_1} \frac{s_\ell(x_\ell) - s_\ell(x_\ell^*)}{L} \\ &\le \frac{n+1}{L}(s_j(x_j) - s_j(x_j^*)) = \frac{n+1}{L}\mathcal{V}(\mathbf{x}) \end{aligned}$$

If $|x_i - x_i^*| > \varepsilon_1$ then

$$\mathcal{V}(\mathbf{x}) > \frac{L\varepsilon_1}{n+1} = \delta_1$$

Thus, for all sufficiently small ε_1, $0 < \varepsilon_1 < r$, there exits $\delta_1 = \frac{L\varepsilon_1}{n} > 0$, such that if $\rho(\mathbf{x}, \mathcal{X}^*) > \varepsilon_1$ then $\mathcal{V}(\mathbf{x}) > \frac{L\varepsilon_1}{n}$.

2) To show that for all sufficiently small $\varepsilon_2 > 0$, there exits a $\delta_2 > 0$, such that for all $\mathbf{x} \in \mathcal{X}$, if $\rho(\mathbf{x}, \mathcal{X}^*) < r$ and $\rho(\mathbf{x}, \mathcal{X}^*) < \delta_2$, then $\mathcal{V}(\mathbf{x}) \le \varepsilon_2$, consider again the following two cases. If $i = j$ then $x_i - x_i^* < 0$, and

$$\mathcal{V}(\mathbf{x}) \le K(x_j^* - x_j) \le K\,|x_i - x_i^*| < K\delta_2 = \varepsilon_2$$

If $i \ne j$, then $x_j - x_j^* < 0$ and $\left|x_j - x_j^*\right| \le |x_i - x_i^*|$, then

$$\begin{aligned} \mathcal{V}(\mathbf{x}) &\le K(x_j^* - x_j) = K\left|x_j - x_j^*\right| \\ &< K\,|x_i - x_i^*| < K\delta_2 = \varepsilon_2 \end{aligned}$$

Therefore, for all sufficiently small $\varepsilon_2 > 0$, there exits $\delta_2 = \frac{\varepsilon_2}{K}$, such that if $\rho(\mathbf{x}, \mathcal{X}^*) < \frac{\varepsilon_2}{K}$ then $\mathcal{V}(\mathbf{x}) \le \varepsilon_2$.

3) Fix a time k. To show that the function $\mathcal{V}$ is non-increasing we must prove that if $\mathbf{x}(0) \in B(\mathcal{X}^*; r)$, then any event in the set $\mathcal{E}$ yields

$$\mathcal{V}(\mathbf{x}(k)) \ge \mathcal{V}(\mathbf{x}(k+1))$$

Suppose that $\mathbf{x}(k) \notin \mathcal{X}^*$ (otherwise $\mathcal{V}(\mathbf{x}(k)) = 0$). Let $s_i(x_i(k)) = \max\{s_\ell(x_\ell(k)) : \ell \in \mathcal{M}_i\}$, that is, $s_i(x_i(k))$ is a local maximum in the neighborhood $\mathcal{N}_i$. We want to show that $s_j(x_j(k) - \gamma_{ji}) < s_i(x_i(k))$ for all $j \in \mathcal{M}_i$. Note that according to eq. (10)

$$\begin{aligned} s_j(x_j(k) - 2\gamma_{ji}) &= s_j\left(\left(\frac{w_j}{w_i}\right)^a x_i(k)\right) \\ &= \frac{w_i^a}{x_i} \\ &= s_i(x_i(k)) \end{aligned}$$

And, because s_j is a non-increasing function

$$s_i(x_i(k)) = s_j(x_j(k) - 2\gamma_{ji}) > s_j(x_j(k) - \gamma_{ji})$$

Since the local maximum is less than or equal to the global maximum, we have

$$\begin{aligned} \max\{s_\ell(x_\ell(k)) : \ell \in \mathcal{M}_i\} &= s_i(x_i(k)) \\ &> s_j(x_j(k) - \gamma_{ji}) \\ &= s_j(x_j(k+1)) \end{aligned}$$

which guarantees that $\mathcal{V}(\mathbf{x}(k)) \ge \mathcal{V}(\mathbf{x}(k+1))$.

4) To show that the function $\mathcal{V}(\mathbf{x}(k)) \to 0$ as $k \to \infty$, note that because of Conditions 1 and 2, $\mathcal{V}(\mathbf{x}(k)) \ge 0$. Any decreasing and lower bounded sequence converges to its greatest lower bound (its infimum). We want to prove that $\inf\{\mathcal{V}(\mathbf{x}(k)) : k \in \mathbb{N}\} = p = 0$ (which is equivalent to saying that $\mathbf{x}(k) \to \mathbf{x}^*$ as $k \to \infty$).
Assume that $p > 0$. Because $\mathcal{V}$ is continuous on $\mathcal{R}$, then using Condition 3, $\mathbf{x}(k)$ converges to the points in the level set

$$\mathcal{L}(p) := \{\mathbf{x} \in \mathcal{X} : \mathcal{V}(\mathbf{x}) = p\}$$

Let $\mathbf{x}(k') \in \mathcal{L}(p)$ characterize an instant in the state of the network where there exists a unique node such that $s_i(x_i(k')) = \max\{s_\ell(x_\ell(k')) : \ell \in \mathcal{N}\}$.
$\mathcal{V}(\mathbf{x}(k')) = p$ if and only if $s_i(x_i(k')) = p + s^*$ which implies that $s_i(x_i(k'))$ is above the equilibrium value s^*. Note that $s_i(x_i(k'))$ must be the local maximum in $\mathcal{N}_i$. Therefore, there exists a node $j \in \mathcal{N}_i$ such that $\gamma_{ji} > 0$. Because $\mathcal{V}$ is a non-increasing function, then

$$\mathcal{V}(\mathbf{x}(k'+1)) < p = \mathcal{V}(\mathbf{x}(k'))$$

which contradicts the fact that p is the infimum of $\mathcal{V}(\mathbf{x}(k))$. Thus $p = 0$. Finally, because $\mathcal{V}$ is continuous on $\mathcal{R}$ we have that $\mathcal{V}(\mathbf{x}(k'))$ equals

$$\max\{s_1(x_1(k')), \ldots, s_{n+1}(x_{n+1}(k'))\} - s^* = 0$$

In other words, $\max\{s_i(x_i(k')) : i \in \mathcal{N}\} = s^*$, which happens if and only if $s_\ell(x_\ell(k')) = s^*$ for all $\ell \in \mathcal{N}$. Thus, $\mathbf{x}(k') = \mathbf{x}^*$ and $\mathcal{V}(\mathbf{x}(k)) \to 0$ as $k \to \infty$.

Therefore, $\mathcal{X}^*$ has a region of asymptotic stability equal to $\mathcal{X}$.

IV. SIMULATIONS

To illustrate the IFD in human decision-making, we present data on the dynamics of a group of 16 students who can choose one of three options, labeled as red, green, and blue. At the end of a round, one participant is randomly selected from the subgroup of students that share the red (green) option and awarded a token. For the blue option seven participants are selected. After fourteen rounds, the participant with the most tokens wins \$25. The experimental platform resembles the virtual environment in [6], [9] and is available at http://jfinke.org/public_html/IFD/index.php.

The sequence, denoted by the sequence $\{\hat{\mathbf{x}}\} = \{\hat{\mathbf{x}}(1), \ldots, \hat{\mathbf{x}}(13)\}$, corresponds to a data trajectory. Each element $\hat{\mathbf{x}} \in \mathcal{R}^2$ where $\hat{x}_i$ denotes the fraction of students choosing option i. Figure 1 shows the ratio of the fraction of students between groups $\frac{x_i}{x_j}$ and the ratio of selected participants $\frac{w_i}{w_j}$ at the end of a round. While the diagonal line represents an IFD with strict matching, the marks indicate the actual dispersal pattern from the data. The empirical measure of the value of a is $\hat{a} = 0.54$.

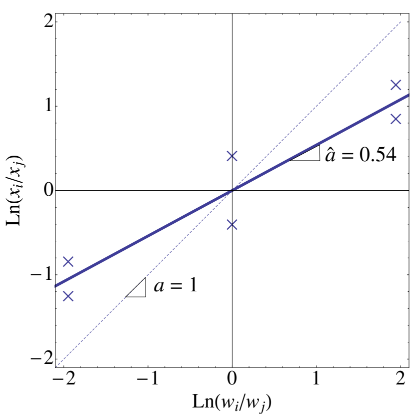


Fig. 1. Group choice ratios from the data.

Figure 2 shows the evolution of the fraction of students choosing each option starting from $\hat{\mathbf{x}}(0) = (\frac{1}{2}, \frac{1}{2})$ (along with the fraction of students selecting the third option). Because seven participants are selected from the subgroup of students that share the blue option, there are on average a total of 10 students who choose this option compare to an average of 3 students for the red (green) option.

To try to capture the evolution of the data with the model presented in Section II, let $q = 16$, $\mathcal{N} = \{1, 2, 3\}$ and $\mathcal{A} = \{\{1,2\}, \{1,3\}, \{2,3\}\}$. To every option $i \in \mathcal{N}$ we associate the marginal utility function (as in eq. (7)), with $w_1 = w_2 = 1$ and $w_3 = 7$. To resemble the level of undermatching observed in the data let $a = 0.54$.

Figure 3 illustrates the outcome of a simulation run. In particular, Figure 3(a) shows the evolution of the marginal utility function associated to each option. The IFD is achieved at a unique state $\mathcal{X}^*$ according to eq. (9). Figure 3(b) shows the fraction of individuals choosing each option (i.e., the state trajectory). Note that because $w_i \neq w_j$, the IFD is

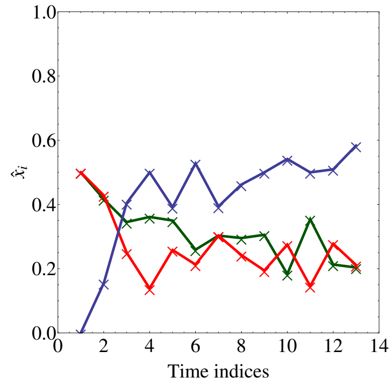


Fig. 2. Fraction of students selecting option i during a single round.

achieved with different number of individuals choosing each option. While the data trajectory (Figure 2) does not reach a stationary value, note that the model captures the qualitative behavior of the dispersal pattern of the students (Figure 3(a)).

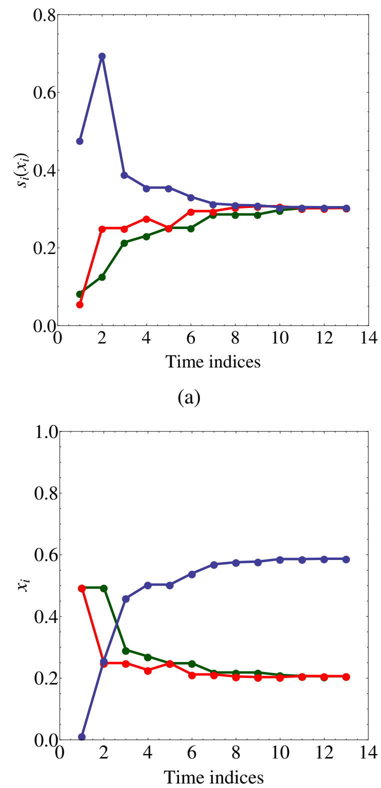
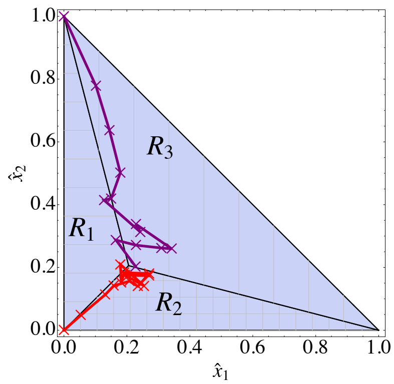
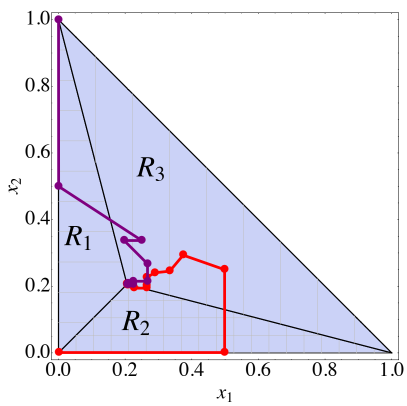


Fig. 3. Dynamics of decision-making simulations leading to the IFD with undermatching; (a) marginal utility values associated to option i; (b) fraction of choices for option $i \in \mathcal{N}$.

Next, consider the set of states for which each option maximizes its associated marginal utility function. Figure 4 depicts these regions denoted by $R_j = \{(x_1, x_2) \in \mathcal{R}^2 : \max_i\{s_i(x_i)\} = s(x_j)\}$. It illustrates how the dynamics evolve across the state space. In particular, Figure 4(a) illustrates how two sample trajectories, starting at initial states in $\mathcal{R}_3$ and $\mathcal{R}_2$, reach a neighborhood close to the predicted IFD value. Figure 4(b) shows two state trajectories with equal initial states. Note that as we approach the IFD, fluctuations in the state trajectory diminish, a consequence of the stability properties of the model.



(a)



(b)

Fig. 4. Regions $R_j = \{(x_1, x_2) \in \mathcal{R}^2 : \max_i\{s_i(x_i)\} = s(x_j)\}$; (a) data trajectory; (b) state trajectory.

Finally, we illustrate how the tendency of individuals to switch between options seems to affect the resulting degree of matching. Figure 5 shows the mean value of the sensitivity parameter a as a function of the total number of individuals changing options. Error-bars indicate one standard deviation for 30 simulations runs. It suggests that as the reallocation of choices increases, the value of the sensitivity parameter increases as well.

V. CONCLUSIONS

The proposed model resembles the outcome of decision-making mechanisms leading to an IFD-like allocation with undermatching. This allocation pattern implies that individuals have an equal probability of success according to both social and environmental determinants (regardless of the initial distribution of choices). Simulations show that when undermatching is present the ratios representing the distribution of individuals is less extreme than the ratios representing the distribution of resources, i.e., individuals tend to distribute their choices in such a way that fewer individuals choose the options with the most resources. To capture the effect of loss aversion on the IFD (i.e., the impact of being more displeased with losses than with equivalent gains) is an important direction for future research.

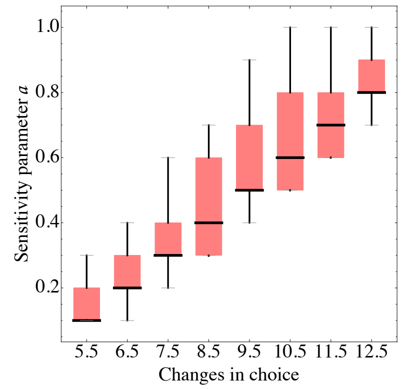


Fig. 5. Number of individuals changing options. Each point represents 30 simulation runs.

VI. ACKNOWLEDGMENTS

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