2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Local Requirements for Optimal Distribution of Heterogeneous Agents

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Abstract— **This paper introduces an analytical framework for the study of a generic distribution problem where a group of heterogeneous agents intend to divide themselves into various subgroups without any form of global information-sharing or centralized decision-making. Subgroups are associated to mathematical functions that capture the marginal utilities of performing tasks, each satisfying the law of diminishing returns. We prove that under generic local requirements a stable agent distribution representing a Nash equilibrium can be achieved, and show via Monte Carlo simulations how the proposed set of rules performs under varying constraints on information flow and degrees of cooperation.**

I. INTRODUCTION

The combination of qualities that form an individual's distinctive skills underlies the problem of optimal team formation (i.e., the division of a group into subgroups). It reflects how individuals behave and interrelate with one another, akin or not [1]. In companies, for instance, individuals belonging to a design team creates innovative ideas for improving customer satisfaction. Members of a management team devote their efforts towards organizing and distributing resources (personnel, money, materials and other assets), while members of an implementation team develop the end products. To coordinate these different teams, companies often rely on hierarchical structures with elements of both central control and divisional autonomy that determine how individuals with complementary skills interact within and across interdisciplinary groups [2].

In biology, where integration and coordination of different types of individuals are key for survival, colonies exhibit under uncertainty a highly decentralized organizational structure [3]. Insects divide their labor force into teams which perform tasks that require different expertise, minimizing the time required to complete them and withstanding adverse environmental conditions (e.g., consider how honey bees search for nest sites [4], wasps store wood [5], or ants collect food [6]).

All social species have a division of labor. Trying to imitate some of nature's remarkable designs, scientists and engineers devise large-scale technological systems that efficiently exploit coordination between agents of different kinds (heterogeneous in character or content). Such technological systems include multiple agents trying, for instance, to maintain a specified formation to fight fires, survey large areas, or transfer numerous goods [7]-[9]. While the ability of homogeneous agents to achieve these objectives is often limited, if not impossible, the design of processes that control the dynamics of heterogeneous agents remains -in many contexts- an open challenge.

In all of the above studies, specialization and distributed decision-making are key in dealing with agents, particularly when the total utility designated to the group rests on the law of diminishing returns [10]. The law of diminishing returns teaches that the marginal utility of a subgroup starts to progressively decrease as the number of agents associated to that subgroup increases. In other words, for a given task, the addition of an agent (of some type) yields smaller increments in utility [11]. Under this assumption, an optimal distribution of the group is reached (i.e., a Nash equilibrium that maximizes group utility) when all subgroups have the same marginal utility [12].

Our work here develops an analytical framework that allows us to describe local requirements which lead to an optimal distribution of heterogeneous agents. The proposed framework is generic and may be of interest in different scenarios in which self-organized behavior emerges among agents that are associated to predefined tasks (e.g., in social or technological systems). It explores the trade-off between the optimal distribution of heterogeneous agents and their optimal level of cooperation.

The remaining sections are organized as follows: Section II states the problem to solve and introduces the model. Section III presents sufficient conditions for a group of heterogeneous agents to reach the optimal distribution. Our analytical results in Section IV extend the work in [11] by relaxing the information flow constraints among agents. Section V provides Monte Carlo simulations that explore the link between cooperation, settling time, and information-sharing.

II. THE BASIC PROBLEM

Let a node represent an activity, service, task, or subgroup of agents; nodes belong to a set N, indexed from 1 to n. An agent represents a resource or unit of supply and may be of various types; agent types belong to a set M, indexed from 1 to m. We assume that the amount of agents of each type is large enough to be appropriately represented by a continuous variable (as in [11], [13]). Let $\mathcal{R} = [0, \infty)^m$ be the space of all combinations of available agents and let $r_i = [r_{i1}, ..., r_{im}]^\top \in \mathcal{R}$ be the distribution of agents at node i (e.g., the amount of agents of each type assigned to a particular activity), where $r_{i\ell}$ represents the amount of agents of type ℓ. Let $\Delta_c \subset \mathcal{R}^n$ denote the $m(n-1)$ dimensional simplex defined by the equality constraint $\sum_{i=1}^{n} r_i = c$, where c is a vector $[c_1, ..., c_m]^\top \in \mathcal{R}$ and c_ℓ denotes the available amount of agents of type ℓ. The profitability of having a distribution of agents r_i at node i is given by the utility function $f_i : \mathcal{R} \to [0, \infty)$ and the total utility function is defined by $f : \mathcal{R}^n \to [0, \infty)$, $f(r) = \sum_{i=1}^{n} f_i(r_i)$, where $r = [r_1^\top, ..., r_n^\top]^\top$ represent the states of the system. The objective is to identify local requirements that allow us to solve the following optimization problem

$$\text{maximize } f(r), \text{ subject to } r \in \Delta_c. \tag{1}$$

We require that each utility function f_i satisfies the following three assumptions, common in economic theory [14]. First, each function f_i is continuously differentiable on $\mathcal{R}$. Second, an increase in utility must satisfy the law of diminishing returns expressed as

$$\frac{f_i(r_i + u_\ell h_\ell) - f_i(r_i)}{u_\ell} > \frac{f_i(r_i + w_\ell h_\ell) - f_i(r_i)}{w_\ell} \tag{2}$$

where $r_i \in \mathcal{R}$, $w_\ell > u_\ell > 0$ are some amount of agents of type $\ell \in M$, and $h_\ell \in \mathcal{R}$ is a vector with one in the ℓth row and zeros otherwise. In other words, we assume a decreasing average returns with respect to (w.r.t.) increasing magnitudes of agent additions. Third, an increase of agents at a node must always increase the utility associated to that node and satisfy the bounds

$$0 < \frac{f_i(r_i + v_\ell h_\ell) - f_i(r_i)}{v_\ell} < \infty \tag{3}$$

where $v_\ell > 0$, $\ell \in M$.

Note that (2) and (3) consider a particular case of the law of diminishing returns. Under these assumptions, the partial derivative of f_i w.r.t. agents of type $\ell \in M$, that is the marginal utility w.r.t. agents of type ℓ, denoted as $s_{i\ell}$, satisfies

$$-a_\ell \leq \frac{s_{i\ell}(x_i) - s_{i\ell}(y_i)}{x_{i\ell} - y_{i\ell}} \leq -b_\ell \tag{4}$$

for any $x_i, y_i \in \mathcal{R}$, $x_{i\ell} \neq y_{i\ell}$, and some constants $0 < b_\ell \leq a_\ell$ defined for each type of agent $\ell \in M$. Because of the assumptions on f_i, the functions $s_{i\ell}$ are continuous on $\mathcal{R}$, strictly decreasing, and non-negative (see Appendix A for details).

Since each utility function f_i is strictly concave, then f is strictly concave on $\mathcal{R}^n$ and the optimal point $r^\star$ that satisfies (1) is unique and belongs to a set $\Delta_c^\star$ which is equal to

$$\{r \in \Delta_c | \, \forall j, k \in N, \, \forall \ell \in M, \, s_{j\ell}(r_j) < s_{k\ell}(r_k) \Rightarrow r_{j\ell} = 0\} \tag{5}$$

In other words, when $r \in \Delta_c^\star$, it must be the case that if a node j has a lower marginal utility w.r.t. agents of type ℓ, the optimal distribution has no agents of type ℓ at that node [15].

III. THE MODEL

The interconnection among nodes is described by a network $G = (N, A)$, where $A \subset N \times N$. If $(j, k) \in A$, agents at node j can sense information about node k and can move to k. Let $p(j) = \{k : (j, k) \in A\}$ denote the set of neighboring nodes of j. We assume that agents move at time index $t = 0, 1, 2, ...$, driven by the asynchronous occurrence of discrete events.

For a node $j \in N$ and $k \in p(j)$, let $e_{u_{j\ell}}^{j\to k}(t)$ denote a movement of an amount $u_{j\ell}$ of agents of type $\ell \in M$ from node j to node k at time t. Let $e_{u_j}^{j\to k}(t)$ denote the set of all possible movements (of any type of agent) from node j to k, where $u_j = [u_{j1}, ..., u_{jm}]^\top \in \mathcal{R}$ represents the amount of each type of agent leaving node j. Let $e_{u_j}^{j\to p(j)}(t)$ denote the set of all possible simultaneous movements from node j to its neighboring nodes $p(j)$. Finally, let $\mathcal{E} = \mathcal{P}(\{e_{u_j}^{j\to p(j)}(t)\}) - \{\emptyset\}$ be the set of events of all simultaneous movements between nodes. An event $e(t) \in \mathcal{E}$ is defined as a set where each element represents a movement of an amount of agents of type $\ell \in M$ between two neighboring nodes.

If an event $e(t) \in \mathcal{E}$ occurs at time t, the update of the state of the system is given by $r(t + 1) = g(r(t))$. For the agents of type $\ell \in M$ at node $i \in N$, $g(r(t))$ is defined as

$$\begin{aligned} r_{i\ell}(t+1) = r_{i\ell}(t) \quad &- \sum_{\{k: e_{u_{i\ell}}^{i\to k}(t) \in e(t)\}} u_{i\ell}(t) \\ &+ \sum_{\{j: e_{u_{j\ell}}^{j\to i}(t) \in e(t)\}} u_{j\ell}(t) \end{aligned} \tag{6}$$

To solve (1) we require that the model satisfies the following three assumptions.

Assumption 1 (on the network): The network $G = (N, A)$ is

i. Undirected: $\forall j, k \in N$, $j \in p(k) \leftrightarrow k \in p(j)$.
ii. Connected: $\forall j, k \in N$ there exists a path from node j to node k.

In other words, the network has only one component and if agents of any type can sense (or move) from any node j to k, they can also sense information from node k to j. Assumption 1 places minimum requirements on the sensing topology to be considered (as well as on the possible movement of agents).

Assumption 2 (on the amount of agents): The total amount of agents of each type $\ell \in M$, $c_\ell > c_\ell^\star$, is large enough for nodes to gain some marginal utility w.r.t. each type of agent when $r \in \Delta_c^\star$.

We can calculate the agent threshold $c_\ell^\star$ in terms of the two constants a_ℓ and b_ℓ for $\ell \in M$. Similar to [13], we can show that when Assumption 1 and 2 hold, because all nodes gain some marginal utility w.r.t. each type of agent, $\Delta_c^\star$ can be written as

$$\{r \in \Delta_c | \, \forall j \in N, \, \forall k \in p(j), \, \forall \ell \in M, \, s_{j\ell}(r_j) = s_{k\ell}(r_k)\} \tag{7}$$

In other words, when $r \in \Delta_c^\star$, it must be the case that all nodes have the same marginal utility w.r.t. agents of type ℓ.

Assumption 3 (on the agent decision-making): Agents movements are stochastic, but must satisfy the following rules for $e(t) \in \mathcal{E}$ at time t.

If $s_{j\ell}(r_j(t)) \geq s_{k\ell}(r_k(t))$ $\forall k \in p(j)$, the movement $e_{u_{j\ell}}^{j\to k}(t) \in e(t)$ has $u_{j\ell}(t) = 0$, i.e., no amount of agents of type $\ell \in M$ may leave node j. Otherwise, the movement of agents of type ℓ from a node $j \in N$ satisfy:

i. If $e^{j\to k}_{u_{j\ell}}(t),\ e^{j\to k'}_{u_{j\ell}}(t) \in e(t)$, then $k = k'$; for agents of type $\ell \in M$ at node j there is a unique destination node k, such that

$$s_{k\ell}(r_k(t)) \geq s_{i\ell}(r_i(t)) > s_{j\ell}(r_j(t)), \forall i, k \in p(j) \quad (8)$$

ii. For $e^{j\to k}_{u_{j\ell}}(t) \in e(t)$, the amount of agents of type ℓ that decide to move to node k

$$0 < u_{j\ell}(t) = \phi_\ell[s_{k\ell}(r_k(t)) - s_{j\ell}(r_j(t))] \quad (9)$$

where $\phi_\ell \in (0, 1/2a_\ell]$.

Eq. (8) guarantees that agents try to move to a node that has higher or equal marginal utility than all other neighboring nodes of j. The value of ϕ_ℓ in (9) captures the level of cooperation between the agents of type $\ell \in M$. Note that a_ℓ, the bound on the marginal utility w.r.t. agents of type ℓ, limits the degree of cooperation between agents of type ℓ. In other words, the amount of agents moving from node j to k is bounded by the fastest change in marginal utilities w.r.t. agents of type $\ell \in M$. Assumption 3 restrains the allowable events in the network and bounds the amount of agents that can move between nodes. Assumptions 1-3 allow us to extend the model in [11] by defining local requirements for distributed decision-making strategies that lead to an optimal distribution of heterogeneous agents.

IV. RESULTS

Theorem 1: Suppose Assumptions 1-3 hold. The point $r \in \Delta_c^\star$ is an equilibrium point of the model and has a region of asymptotic stability equal to Δ_c.

Proof:

Let a Lyapunov candidate function be defined as

$$V(r) = \sum_{\ell=1}^{m} \left(\max_i \{s_{i\ell}(r_i)\} - s_{i\ell}(r_i^\star) \right) \quad (10)$$

where $r^\star \in \Delta_c^\star$. In Appendix B, we show that for $r \in \Delta_c$ and the choice of the metric $\rho(r, \Delta_c^\star)$ defined by 30, there exist two constants $\eta_1, \eta_2 > 0$ such that $\eta_1 \rho(r, \Delta_c^\star) \leq V(r) \leq \eta_2 \rho(r, \Delta_c^\star)$. Here, we show that as $t \to \infty$, $V(r(t)) \to 0$.

Note that when $r \in \Delta_c^\star$, $\forall j \in N$, $\forall k \in p(j)$, $\forall \ell \in M$, we know that $s_{j\ell}(r_j) = s_{k\ell}(r_k)$. Because of Assumption 3, agents cannot move between nodes and the associated marginal utilities do not change, i.e., $u_{j\ell} = 0$.

Let $r \notin \Delta_c^\star$. For $e(t) \in \mathcal{E}$ at time t, if $e^{j\to k}_{u_{j\ell}}(t) \in e(t)$, the agents that move away from (arrive at) node j increase (decrease, respectively) its marginal utility. Because Assumption 3 (i) guarantees that the agents of type $\ell \in M$ at node j can only move to a unique destination node at time t, the marginal utility at node j at time $t+1$ is bounded by

$$s_{j\ell}(r_j(t+1)) \leq s_{j\ell}(r_j(t) - u_{j\ell}(t) h_\ell) \quad (11)$$

In other words, the marginal utility of node j is bounded by the amount of agents that leave that node.

Let $\mathcal{K}_\ell(t) = \arg\max_i \{s_{i\ell}(r_i(t))\}$ be the set of nodes with the highest marginal utility w.r.t. agents of type $\ell \in M$ at time t. We want to show that the maximum marginal utility is non-increasing (i.e., there is no node that exceeds the value of marginal utility of the nodes in $\mathcal{K}_\ell(t)$). Consider the movement $e^{j\to k}_{u_{j\ell}}(t)$ of some agents from node j to node k. Using (4) for node k with $x_k = r_k(t) + u_{j\ell}(t) h_\ell$ and $y_k = r_k(t)$, yields

$$\begin{aligned} s_{k\ell}(r_k(t)) - b_\ell u_{j\ell}(t) &\geq s_{k\ell}(r_k(t) + u_{j\ell}(t) h_\ell) \\ &\geq s_{k\ell}(r_k(t)) - a_\ell u_{j\ell}(t) \end{aligned} \quad (12)$$

Accordingly to Assumption 3 (ii), when $e^{j\to k}_{u_{j\ell}}(t) \in e(t)$, we know that

$$s_{k\ell}(r_k(t)) - a_\ell u_{j\ell}(t) \geq s_{j\ell}(r_j(t)) + a_\ell u_{j\ell}(t) \quad (13)$$

And using (4) for node j with $y_j = r_j(t) - u_{j\ell}(t) h_\ell$ and $x_j = r_j(t)$, yields

$$s_{j\ell}(r_j(t)) + a_\ell u_{j\ell}(t) \geq s_{j\ell}(r_j(t) - u_{j\ell}(t) h_\ell) \quad (14)$$

By combining (12), (13) and (14), we get

$$\begin{aligned} s_{k\ell}(r_k(t)) - b_\ell u_{j\ell}(t) &\geq s_{k\ell}(r_k(t) + u_{j\ell}(t) h_\ell) \\ &\geq s_{j\ell}(r_j(t)) + a_\ell u_{j\ell}(t) \\ &\geq s_{j\ell}(r_j(t) - u_{j\ell}(t) h_\ell) \end{aligned} \quad (15)$$

From (11) and (15), we have that

$$\begin{aligned} \max_i \{s_{i\ell}(r_i(t))\} &\geq s_{k\ell}(r_k(t)) \\ &> s_{k\ell}(r_k(t)) - b_\ell u_{j\ell}(t) \\ &\geq s_{j\ell}(r_j(t+1)) \end{aligned} \quad (16)$$

Using (16) and replacing the amount of agents $u_{j\ell}(t)$ that leave the node j (defined in (9)), we know that

$$\begin{aligned} \max_i \{s_{i\ell}(r_i(t))\} &> s_{k\ell}(r_k(t)) - b_\ell \phi_\ell \left[s_{k\ell}(r_k(t)) - s_{j\ell}(r_j(t))\right] \\ &\geq s_{j\ell}(r_j(t+1)) \end{aligned} \quad (17)$$

Finally, since (17) holds for all $j \notin \mathcal{K}_\ell(t)$, we know that the values of marginal utility w.r.t. agents of type $\ell \in M$ in any neighborhood at time $t+1$ do not exceed the actual maximum marginal utility of the nodes in $\mathcal{K}_\ell(t)$.

Furthermore, if $j \in \mathcal{K}_\ell(t)$, accordingly to Assumption 3, there can be no agents leaving that node, so $u_{j\ell}(t) = 0$. Thus, the maximum marginal utility value is non-increasing over time.

Next, we show that after certain number of time steps the maximum marginal utility must decrease. Note that $\forall \ell \in M$, if $|\mathcal{K}_\ell(t)| = n$ then $r \in \Delta_c^\star$. If $r \notin \Delta_c^\star$, accordingly to Assumption 1 (i), there exists some node $k \in \mathcal{K}_\ell(t)$ with a neighboring node $j \in p(k)$ such that $s_{k\ell}(r_k(t)) > s_{j\ell}(r_j(t))$. Accordingly to Assumptions 3, $e^{j\to k}_{u_{j\ell}}(t) \in e(t)$ with $u_{j\ell}(t) > 0$ must occur. For convenience, let $w_{k\ell}(t) = \sum_{\{j : e^{j\to k}_{u_{j\ell}}(t) \in e(t)\}} u_{j\ell}(t)$ be the total amount of agents of type ℓ that arrive at node k from its neighboring nodes $p(k)$ at time t. Because of Assumption 3, we know that $u_{k\ell} = 0$, and using (6) we have that

$$\begin{aligned} s_{k\ell}(r_k(t+1)) &= s_{k\ell}(r_k(t) + w_{k\ell}(t) h_\ell) \\ &\leq s_{k\ell}(r_k(t) + u_{j\ell}(t) h_\ell) \\ &< s_{k\ell}(r_k(t)) \end{aligned} \quad (18)$$

In other words, the marginal utility of node k must decrease at time $t+1$ due to the arrival of agents from $p(k)$.

For any $\ell \in M$, either $|\mathcal{K}_\ell(t+1)| \leq |\mathcal{K}_\ell(t)| - 1$ or $\max_i\{s_{i\ell}(r_i(t))\} > \max_i\{s_{i\ell}(r_i(t+1))\}$ (due to (17) and (18)). Since $|\mathcal{K}_\ell(t)| < n$, after at most n time steps all the highest nodes have been reached by some agents of type $\ell \in M$ and $\max_i\{s_{i\ell}(r_i(t))\} > \max_i\{s_{i\ell}(r_i(t+n))\}$, i.e., the maximum marginal utility value at time $t+n$ must decrease.

Next, fix a time t and let $k \in \mathcal{K}_\ell(t)$. Let $t'' = t+n$ and $\mathcal{J}_\ell(t'') = \{\arg\max_i\{s_{i\ell}(r_i(t))\} : \exists t' \in [t, t''], w_{i\ell}(t') > 0\} \cup \{i \in \mathcal{K}_\ell^{\complement}(t)\}$, i.e., the set of all nodes that were in the set of maxima at time t, where agents arrived between t and t'' (causing a decrease in its marginal utility), plus all nodes that did not belong to the set of maxima $\mathcal{K}_\ell(t)$. Note that the nodes in the set $\mathcal{J}_\ell(t'')$ have a marginal utility below the maximum marginal utility of nodes in $\mathcal{K}_\ell(t)$, i.e.,

$$\max_{i\in\mathcal{J}_\ell(t'')}\{s_{i\ell}(r_i(t')) : t \leq t' \leq t''\} < \max_{k\in\mathcal{K}_\ell(t)}\{s_{k\ell}(r_k(t))\}$$

For a node $k \in \mathcal{K}_\ell(t)$ and $j \notin \mathcal{K}_\ell(t)$, we know that

$$\begin{aligned} &s_{k\ell}(r_k(t)) - \max_{i\in\mathcal{J}_\ell(t'')}\{s_{i\ell}(r_i(t')) : t \leq t' \leq t''\} \\ &\leq s_{k\ell}(r_k(t)) - s_{j\ell}(r_j(t)) \end{aligned} \tag{19}$$

At time t'', when all the nodes in the set $\mathcal{K}_\ell(t)$ have been reached by some agents of type $\ell \in M$, (17) and (19) yield

$$\begin{aligned} &s_{k\ell}(r_k(t)) - b_\ell\phi\left[s_{k\ell}(r_k(t)) - \max_{i\in\mathcal{J}_\ell(t'')}\{s_{i\ell}(r_i(t')) : t \leq t' \leq t''\}\right] \\ &\geq s_{j\ell}(r_j(t'')) \end{aligned} \tag{20}$$

where $\phi = \min_\ell\{\phi_\ell\}$ represents the lowest degree of cooperation between all types of agents. Because (20) is valid for any node $j \in N$, looking at the maximum marginal utility at time t'', we know that

$$\begin{aligned} 0 &> -b_\ell\phi\left[s_{k\ell}(r_k(t)) - \max_{i\in\mathcal{J}_\ell(t'')}\{s_{i\ell}(r_i(t')) : t \leq t' \leq t''\}\right] \\ &\geq \max_i\{s_{i\ell}(r_i(t''))\} - \max_i\{s_{i\ell}(r_i(t))\} \end{aligned} \tag{21}$$

Finally, let

$$\begin{aligned} \Delta V(r(t)) &\triangleq V(r(t'')) - V(r(t)) \\ &= \sum_{\ell=1}^{m}\left(\max_i\{s_{i\ell}(r_i(t''))\} - s_{i\ell}(r_i^\star)\right) \\ &\quad - \sum_{\ell=1}^{m}\left(\max_i\{s_{i\ell}(r_i(t))\} - s_{i\ell}(r_i^\star)\right) \\ &= \sum_{\ell=1}^{m}\left(\max_i\{s_{i\ell}(r_i(t''))\} - \max_i\{s_{i\ell}(r_i(t))\}\right) \end{aligned} \tag{22}$$

Using (21) in (22), yields

$$\begin{aligned} &\Delta V(r(t)) \\ &\leq -\sum_{\ell=1}^{m}\phi b_\ell\left[\max_k\{s_{k\ell}(r_k(t))\} - \max_{i\in\mathcal{J}(t'')}\{s_{i\ell}(r_i(t')) : t \leq t' \leq t''\}\right] \end{aligned} \tag{23}$$

Eq. (23) shows that $V(r(t))$ is a non-increasing function over time and since V is bounded from below there must exist a scalar $q \geq 0$ such that $V(r(t)) \to q$ as $t \to \infty$. Assume $q > 0$. Because V is continuous, $r(t)$ converges to an ω-limit set $\Omega(r(t))$ which is a subset of the level $S_q = \{r \in \Delta_c : V(r) = q\}$. Take any point $\overline{r} \in \Omega(r(t))$, so that $V(\overline{r}) = q > 0$. We need to show that there exists a time index t such that $V(r(t)) < 0$, which contradicts the fact that $0 \leq V(r)$ for all $r \in \Delta_c$ and thus proves that $q = 0$ and $r(t) \to r^\star$.

Because $\overline{r} \in \Omega(r(t))$ there exists, by definition, an infinite sequence of times $T \subset \mathbb{N}$ such that $\{r(t)\}_{t\in T} \to \overline{r}$. Since all the marginal utility functions are continuous, it is also the case that

$$\{s_{k\ell}(r_k(t)) - s_{j\ell}(r_j(t))\}_{t\in T} \to s_{k\ell}(\overline{r}_k) - s_{j\ell}(\overline{r}_j) \in \mathbb{R} \tag{24}$$

For some agents of type $\ell \in M$, define the set $A_\ell = \arg_{(k,j)}\{\max_k\{s_{k\ell}(\overline{r}_k)\} - \max_j\{s_{j\ell}(\overline{r}_j) : j \notin \mathcal{K}_\ell\}\}$, i.e., the set of all the node combinations having a positive difference in their marginal utilities. For convenience, let $\alpha_\ell(t')$ represent the pair (k, j) such that $k \in \mathcal{K}_\ell(t)$, $j = \arg\max_{i\in\mathcal{J}_\ell(t'')}\{s_{i\ell}(r_i(t')) : t \leq t' \leq t''\}$, and $t' = \arg_{t'}\max_{i\in\mathcal{J}_\ell(t'')}\{s_{i\ell}(r_i(t')) : t \leq t' \leq t''\}$. As a consequence of (24), there must exist a time index τ such that $\alpha_\ell(t') \in A_\ell$ for all $t \in T \cap [\tau, \infty) = T_1$. Take $\overline{\alpha}_\ell \in A_\ell$ such that $T_2 = \{t' \in T_1 : \alpha_\ell(t') = \overline{\alpha}_\ell\}$ is an infinite set (let the indices k and j be those defined by $\overline{\alpha}_\ell$). Since $\overline{r} \neq r^\star$ and because the choice of $\overline{\alpha}_\ell$ it is the case that $s_{k\ell}(\overline{r}_k) - s_{j\ell}(\overline{r}_j) = \delta_\ell > 0$, thus $\{\max_k\{s_{k\ell}(r_k(t))\} - \max_{i\in\mathcal{J}_\ell(t'')}\{s_{i\ell}(r_i(t')) : t \leq t' \leq t''\}\}_{t\in T_2} \to \delta$, where $\delta = \min_\ell\{\delta_\ell\}$. Accordingly,

$$\begin{aligned} \Delta V(r(t)) &\leq -\sum_{\ell=1}^{m}\phi b_\ell\left[\max_k\{s_{k\ell}(r_k(t))\} - \max_{i\in\mathcal{J}(t'')}\{s_{i\ell}(r_i(t')) : t \leq t' \leq t''\}\right] \\ &\leq -\sum_{\ell=1}^{m}\phi b_\ell\delta < 0 \end{aligned} \tag{25}$$

Since $V(r(t))$ is non-increasing at each time t, by using (25)

$$\sum_{t\in T_2}\Delta V(r(t)) \leq -\sum_{t\in T_2}\min\{b_\ell\}\phi\delta < 0 \tag{26}$$

is unbounded. It must be that at some finite time index t, there will be the case that $V(r(t)) < 0$. Since $V(r)$ is non-negative for all $r \in \Delta_c$, we arrive at a contradiction. Hence, $q = 0$ and the system has an optimal point, $r(t) \to r^\star$ which is asymptotically stable in the region Δ_c.

V. SIMULATIONS

This section presents the dynamics of heterogeneous agents starting from an initial random distribution across a network of ten nodes. The left plot in Figure 1 shows how the proposed local decision-making leads to the optimal distribution defined by (7). All the marginal utilities converge to a same value. The right plot in Figure 1 shows the dynamics in Δ_c of both types of agents at a randomly selected node. Fluctuations represent agents movements going back and forth before reaching the equilibrium $\Delta_c^\star$.

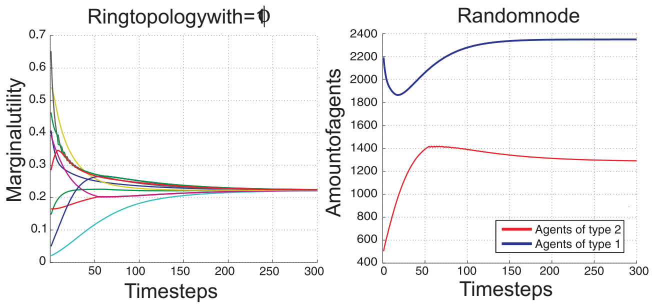

Fig. 1. Agent dynamics satisfying the proposed local decision rules.

Figure 2 shows the effect of network density (i.e., the proportion of edges relative to the total number possible) on the time needed to reach $\Delta_c^\star$ within a 2% margin of error (the settling time, t_s). Employing topologies with high density does not necessarily affect t_s. Note that only for very low densities, t_s increases (for density values greater than .44, the settling time decreases only slightly). Moreover, as group cooperation (ϕ_ℓ) decreases, the mean and the upper bounds on the standard deviation increase (which is consistent with (20)), especially in low densities networks with higher constraints on information flow.

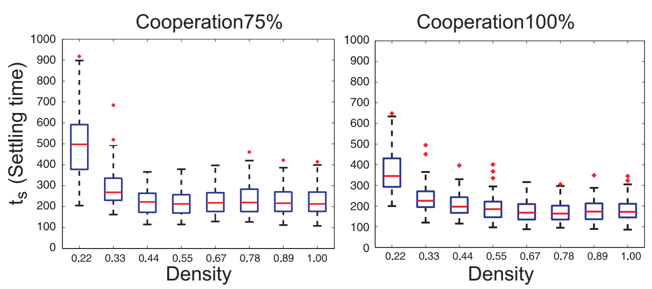

Fig. 2. Settling time vs. network density

Finally, Figure 3 shows the effect of group cooperation on t_s (for convenience we rescale the cooperation level according to the bounds on the set of utility functions). Note that increasing the cooperation level reduces t_s. Note also that for a fixed density (say a density of .55), t_s remains constant for a wide range of group cooperation (from 50% to 100%).

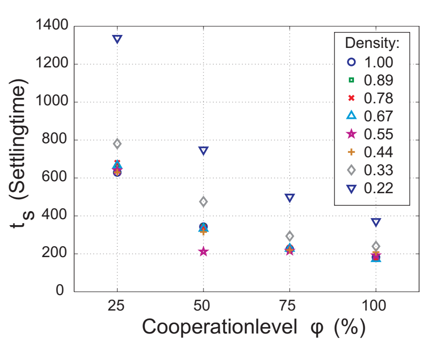

Fig. 3. Settling time for different network densities under varying levels of cooperation between agents.

Figure 3 suggests that increasing the proportion of links relative to the total number possible (say above .33) may be a lost effort if we are trying to increase the rate at which agents reach the optimal point. (Of course, a network with higher density is more robust in the sense that no particular link may be crucial for the network to satisfy Assumption 1 and for the agents to reach convergence).

APPENDIX A

Here we prove that (4) is a special case of the law of diminishing returns expressed in (2). Using (3), we know that

$$\infty > \frac{f_i(r_i+u_\ell h_\ell)-f_i(r_i)}{u_\ell} - \frac{f_i(r_i+w_\ell h_\ell)-f_i(r_i)}{w_\ell} > 0 \quad (27)$$

where $w_\ell > u_\ell > 0$ are the amounts of agents of type $\ell \in M$. Since the utility functions f_i are continuously differentiable and (27) is valid for any $w_\ell > u_\ell > 0$, if we let both quantities tend to zero, we have that

$$\infty > s_{i\ell}(r_i + u_\ell h_\ell) - s_{i\ell}(r_i + w_\ell h_\ell) > 0 \quad (28)$$

where $s_{i\ell}$ is the partial derivative of f_i w.r.t. agents of type $\ell \in M$. Therefore, since $w_\ell \neq u_\ell$, for each type of agents there exist two constants $\infty > a_\ell \geq b_\ell > 0$, such that

$$\begin{aligned} -a_\ell(w_\ell - u_\ell) &\leq s_{i\ell}(r_i + w_\ell h_\ell) - s_{i\ell}(r_i + u_\ell h_\ell) \\ &\leq -b_\ell(w_\ell - u_\ell) \end{aligned} \quad (29)$$

Therefore, letting $x_i = r_i + w_\ell h_\ell$ and $y_i = r_i + u_\ell h_\ell$, we have (4). Thus, if (2) and (3) are satisfied so is (4). (Here, we consider a special case of the law of diminishing returns, where the rate of change of the marginal returns is bounded by constants).

APPENDIX B

Bounds on the Lyapunov Function

Let $r^\star = [r_1^{\star\top}, r_2^{\star\top}, \ldots, r_n^{\star\top}]^\top$ be the optimal distribution of agents in each node (when $r \in \Delta_c^\star$), and choose the metric

$$\rho(r, \Delta_c^\star) = \sum_{\ell=1}^{m} \rho_\ell(r, \Delta_c^\star) \quad (30)$$

where

$$\rho_\ell(r, \Delta_c^\star) = \max_i \{|r_{i\ell} - r_{i\ell}^\star|\} \quad (31)$$

Note that for $r \in \Delta_c^\star$ it must be the case that $\forall \ell \in M$, $\forall k \in N$ $\max_i\{s_{i\ell}(r_i)\} = s_{k\ell}(r_k^\star)$, i.e., there is no node with higher marginal utility than others and $V(r^\star) = 0$. Moreover, accordingly to the definition of the metric, $\rho(r^\star, \Delta_c^\star) = 0$. Thus if $r \in \Delta_c^\star$ $\eta_1 \rho(r, \Delta_c^\star) \leq V(r^\star) \leq \eta_2 \rho(r, \Delta_c^\star)$ is satisfied for any constants $\eta_1, \eta_2 > 0$.

For $r \notin \Delta_c^\star$, there must exists some agents of type $\ell \in M$ such that $r_{i\ell} \neq r_{i\ell}^\star$ and some node $j = \arg\max_i\{|r_{i\ell} - r_{i\ell}^\star|\}$. Applying (4) to node j with $x_j = r_j$ and $y_j = r_j^\star$, yields

$$b_\ell \rho_\ell(r, \Delta_c^\star) \leq |s_{j\ell}(r_j) - s_{j\ell}(r_j^\star)| \quad (32)$$

One of two cases must be true. First, if $s_{j\ell}(r_j) - s_{j\ell}(r_j^\star) > 0$, we have that

$$\begin{aligned} b_\ell \rho_\ell(r, \Delta_c^\star) &\leq s_{j\ell}(r_j) - s_{j\ell}(r_j^\star) \\ &\leq \max_i \{s_{i\ell}(r_i)\} - s_{i\ell}(r_i^\star) \end{aligned} \tag{33}$$

In the second case, if $s_{j\ell}(r_j) - s_{j\ell}(r_j^\star) < 0$, we have that

$$\begin{aligned} b_\ell \rho_\ell(r, \Delta_c^\star) &\leq s_{j\ell}(r_j^\star) - s_{j\ell}(r_j) \\ &\leq s_{i\ell}(r_i^\star) - \min_i \{s_{i\ell}(r_i)\} \end{aligned} \tag{34}$$

Because the total number of agents is constrained, we know that the amount of agents exceeding the optimal amount at some nodes is the same amount of agents that other nodes need. Next, let A_{out}^ℓ be the minimum amount of agents of type $\ell \in M$ at a node with the minimum marginal utility that must leave the node

$$A_{out}^\ell = \frac{s_{i\ell}(r_i^\star) - \min_i \{s_{i\ell}(r_i)\}}{a_\ell} \tag{35}$$

Similarly, let A_{in}^ℓ be the maximal amount of agents of type $\ell \in M$ that may be needed at some nodes

$$A_{in}^\ell = (n-1) \frac{\max_i \{s_{i\ell}(r_i)\} - s_{i\ell}(r_i^\star)}{b_\ell} \tag{36}$$

For agents of type $\ell \in M$, we know that

$$A_{out}^\ell \leq \sum_{\{i: r_{i\ell} > r_{i\ell}^\star\}} (r_{i\ell} - r_{i\ell}^\star) = \sum_{\{i: r_{i\ell} < r_{i\ell}^\star\}} (r_{i\ell}^\star - r_{i\ell}) \leq A_{in}^\ell \tag{37}$$

Using (35), (36) and (37), we have that

$$\frac{s_{i\ell}(r_i^\star) - \min_i \{s_{i\ell}(r_i)\}}{a_\ell} \leq (n-1) \frac{\max_i \{s_{i\ell}(r_i)\} - s_{i\ell}(r_i^\star)}{b_\ell} \tag{38}$$

By combining (33), (34) and (38), we know

$$\frac{b_\ell^2 \rho(r, \Delta_c^\star)}{a_\ell n} \leq \max_i \{s_{i\ell}(r_i)\} - s_{i\ell}(r_i^\star) \tag{39}$$

Therefore, using (39) in the Lyapunov function expressed in (10) for everyone of types of agent accordingly to the definition of $\rho_\ell(r, \Delta_c^\star)$, we have that

$$V(r) = \sum_{\ell=1}^m \max_i \{s_{i\ell}(r_i)\} - s_{i\ell}(r_i^\star) \geq \rho(r, \Delta_c^\star) \sum_{\ell=1}^m \frac{b_\ell^2}{a_\ell n} \tag{40}$$

Thus, if we let $\eta_1 = \min_\ell \{b_\ell^2 / a_\ell n\}$, $V(r) \geq \eta_1 \rho(r, \Delta_c^\star)$.

Finally, we show that for $r \notin \Delta_c^\star$, there exist a bound for the Lyapunov function such that $V(r) \leq \eta_2 \rho(r, \Delta_c^\star)$.

If $r \notin \Delta_c^\star$, there exist some agents of type $\ell \in M$ such that $r_{i\ell} \neq r_{i\ell}^\star$. Applying (4) to a node $k = \arg\max_i \{s_{i\ell}(r_i)\}$ yields

$$\begin{aligned} 0 < \left| \frac{\max_i \{s_{i\ell}(r_i)\} - s_{k\ell}(r_k^\star)}{\max_i \{|r_{i\ell} - r_{i\ell}^\star|\}} \right| &\leq \frac{|\max_i \{s_{i\ell}(r_i)\} - s_{k\ell}(r_k^\star)|}{|r_{k\ell} - r_{k\ell}^\star|} \\ &\leq a_\ell \end{aligned} \tag{41}$$

Similarly, taking $j = \arg\min_i \{s_{i\ell}(r_i)\}$ in (4)

$$\begin{aligned} 0 < \left| \frac{s_{j\ell}(r_j^\star) - \min_i \{s_{i\ell}(r_i)\}}{\max_i \{|r_{i\ell} - r_{i\ell}^\star|\}} \right| &\leq \frac{|s_{j\ell}(r_j^\star) - \min_i \{s_{i\ell}(r_i)\}|}{|r_{j\ell} - r_{j\ell}^\star|} \\ &\leq a_\ell \end{aligned} \tag{42}$$

Adding (41) and (42), we have that

$$\begin{aligned} &\frac{|\max_i \{s_{i\ell}(r_i)\} - s_{kj}(r_k^\star)| + |s_{j\ell}(r_j^\star) - \min_i \{s_{i\ell}(r_i)\}|}{\max_i \{|r_{i\ell} - r_{i\ell}^\star|\}} \\ &\leq 2a_\ell \end{aligned} \tag{43}$$

Because at the optimal point $r^\star$, $s_{k\ell}(r_k^\star) = s_{j\ell}(r_j^\star)$, using the inequality $|a-b| \leq |a-c| + |c-b|$ in (43), yields

$$\max_i \{s_{i\ell}(r_i)\} - \min_i \{s_{i\ell}(r_i)\} \leq 2a_\ell \rho_\ell(r, \Delta_c^\star) \tag{44}$$

Therefore, using (44) and the definition of $V(r)$ and $\rho(r, \Delta_c^\star)$

$$\begin{aligned} V(r) &\leq \sum_{\ell=1}^m \left(\max_i \{s_{i\ell}(r_i)\} - \min_i \{s_{i\ell}(r_i)\} \right) \\ &\leq 2\rho(r, \Delta_c^\star) \sum_{\ell=1}^m a_\ell \end{aligned} \tag{45}$$

Thus, if $\eta_2 = 2m \max_\ell \{a_\ell\}$, $V(r) \leq \eta_2 \rho(r, \Delta_c^\star)$.

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