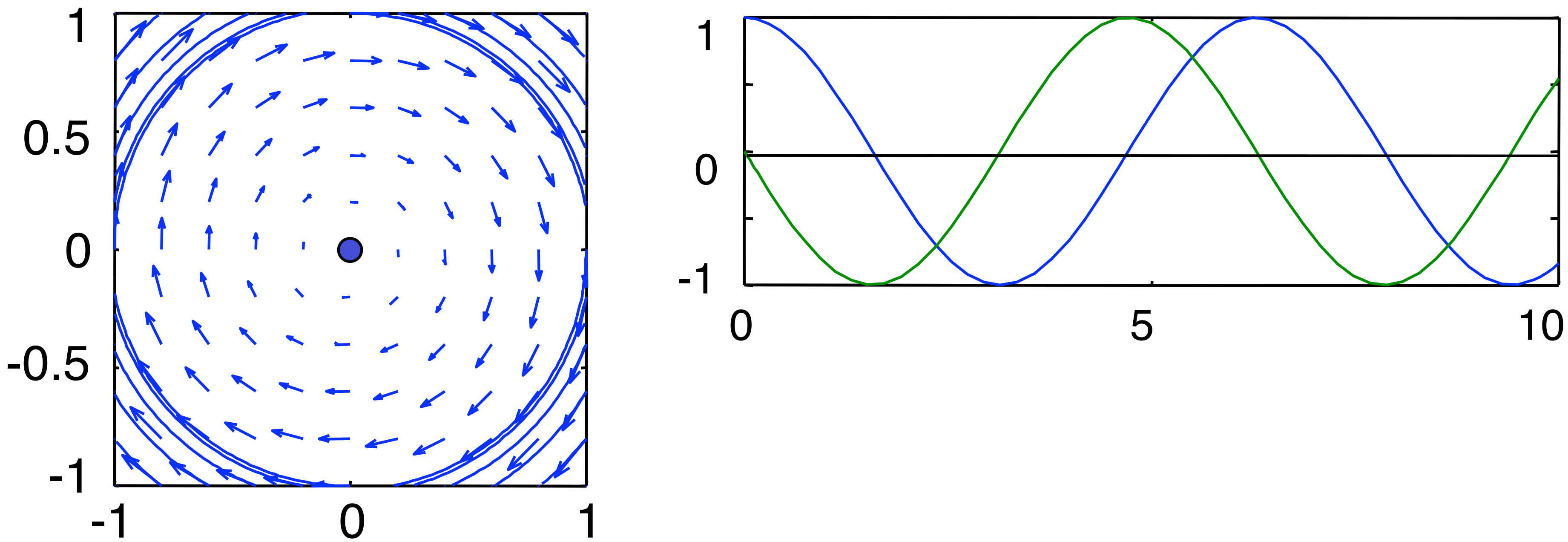
Lecture 6

Review

- Stability for continuous systems
  - Stability in the sense of Lyapunov
  - Asymptotic stability
  - Exponential stability
- Lyapunov’s second method
- Stability of Linear systems

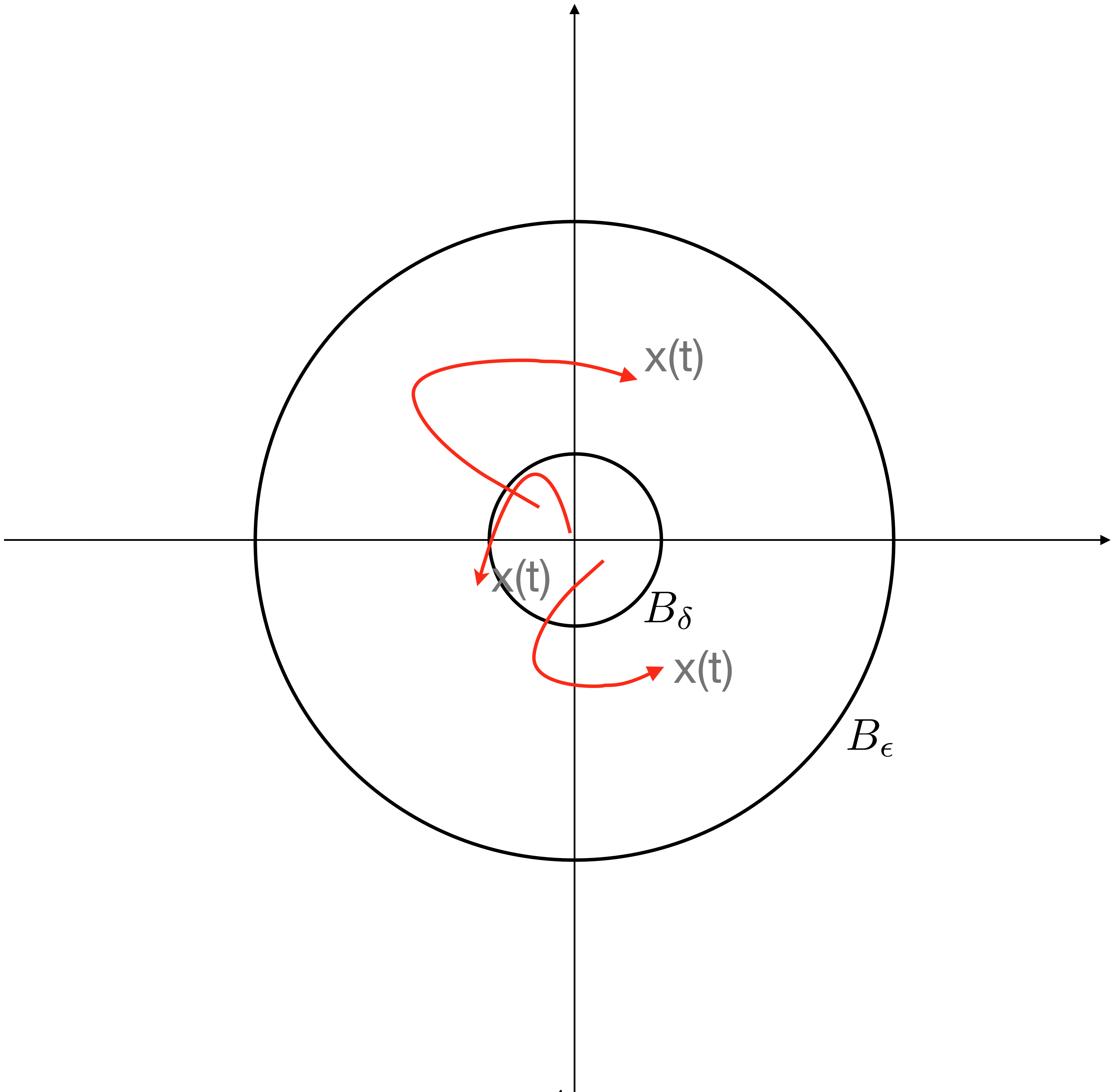
Stable (in the sense of Lyapunov)



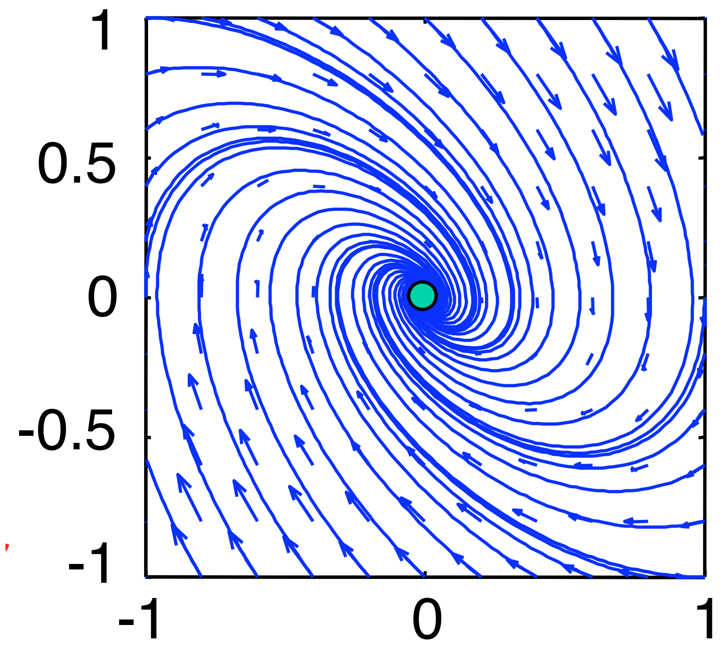
For every $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that, if $\|x(0) - x_e\| < \delta$, then $\|x(t) - x_e\| < \epsilon$, for every $t \geq 0$.

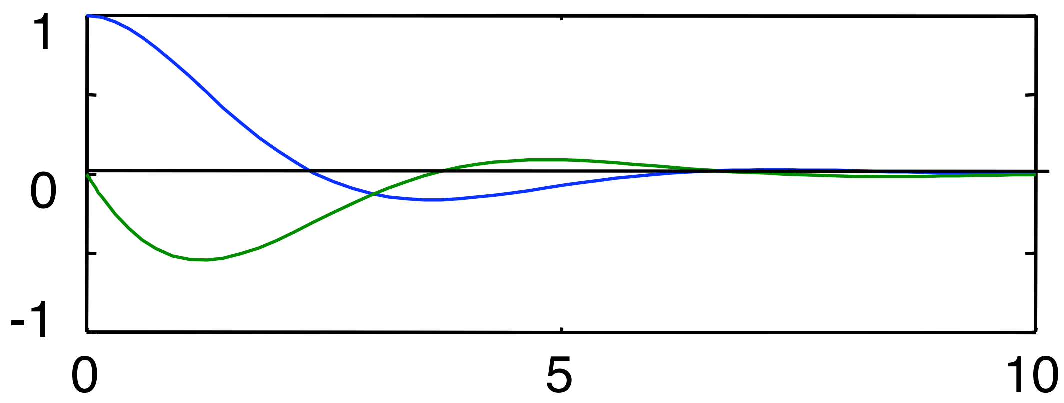
e.p. (equilibrium point) is a center

Switched Systems



Asymptotically Stable



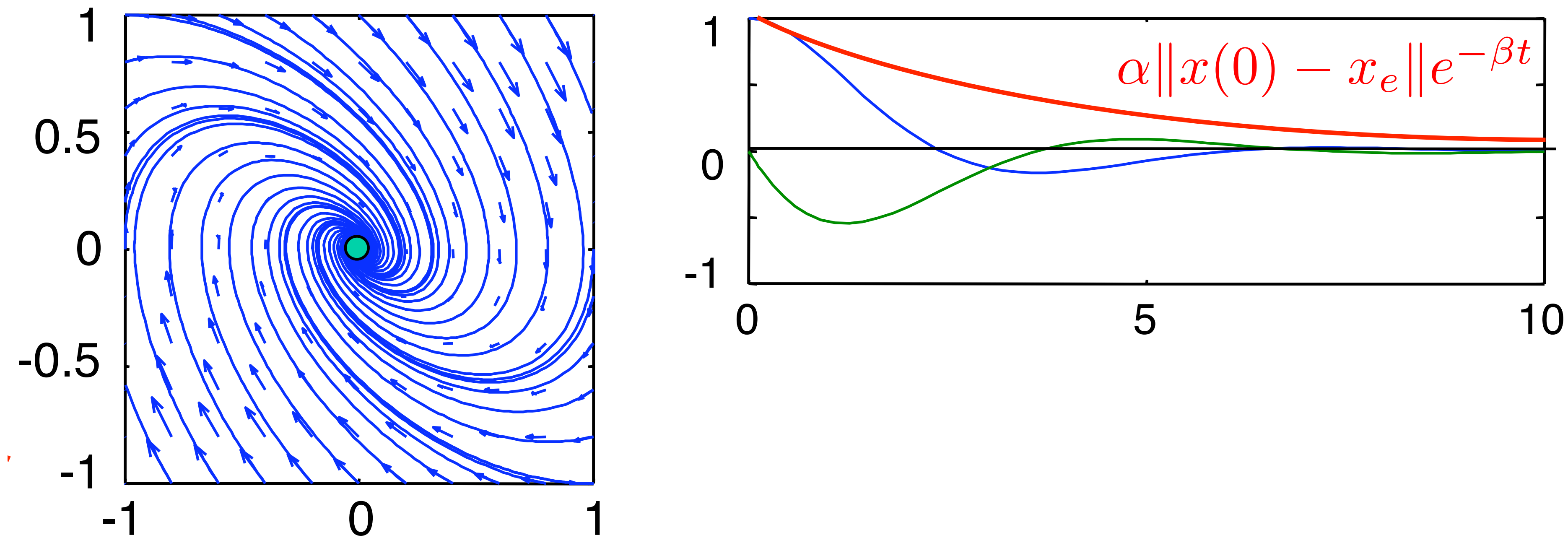


1. Lyapunov stable

2. There exists $\delta > 0$ such that if $\|x(0) - x_e\| < \delta$, then $\lim_{t\to\infty} \|x(t) - x_e\| = 0$.

e.p. is an attractor or sink

Exponentially stable



1. Asymptotically stable

2. There exist $\alpha, \beta, \delta > 0$ such that if $\|x(0) - x_e\| < \delta$, then $\|x(t) - x_e\| \leq \alpha\|x(0) - x_e\|e^{-\beta t}$, for $t \geq 0$.

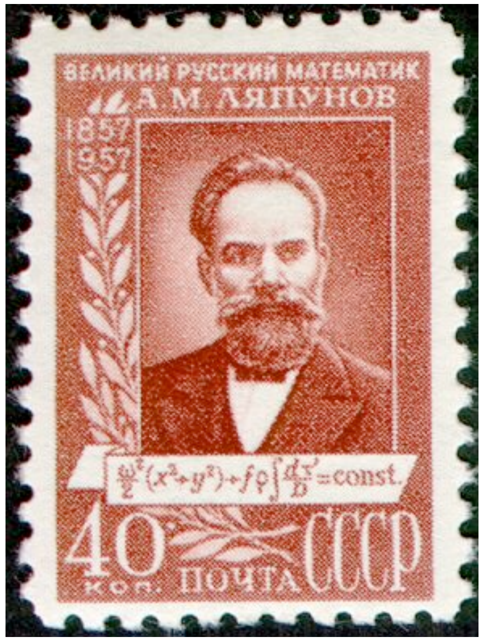
Any e.p. that is exponentially stable systems is also asymptotically stable.

Lyapunov Second Method

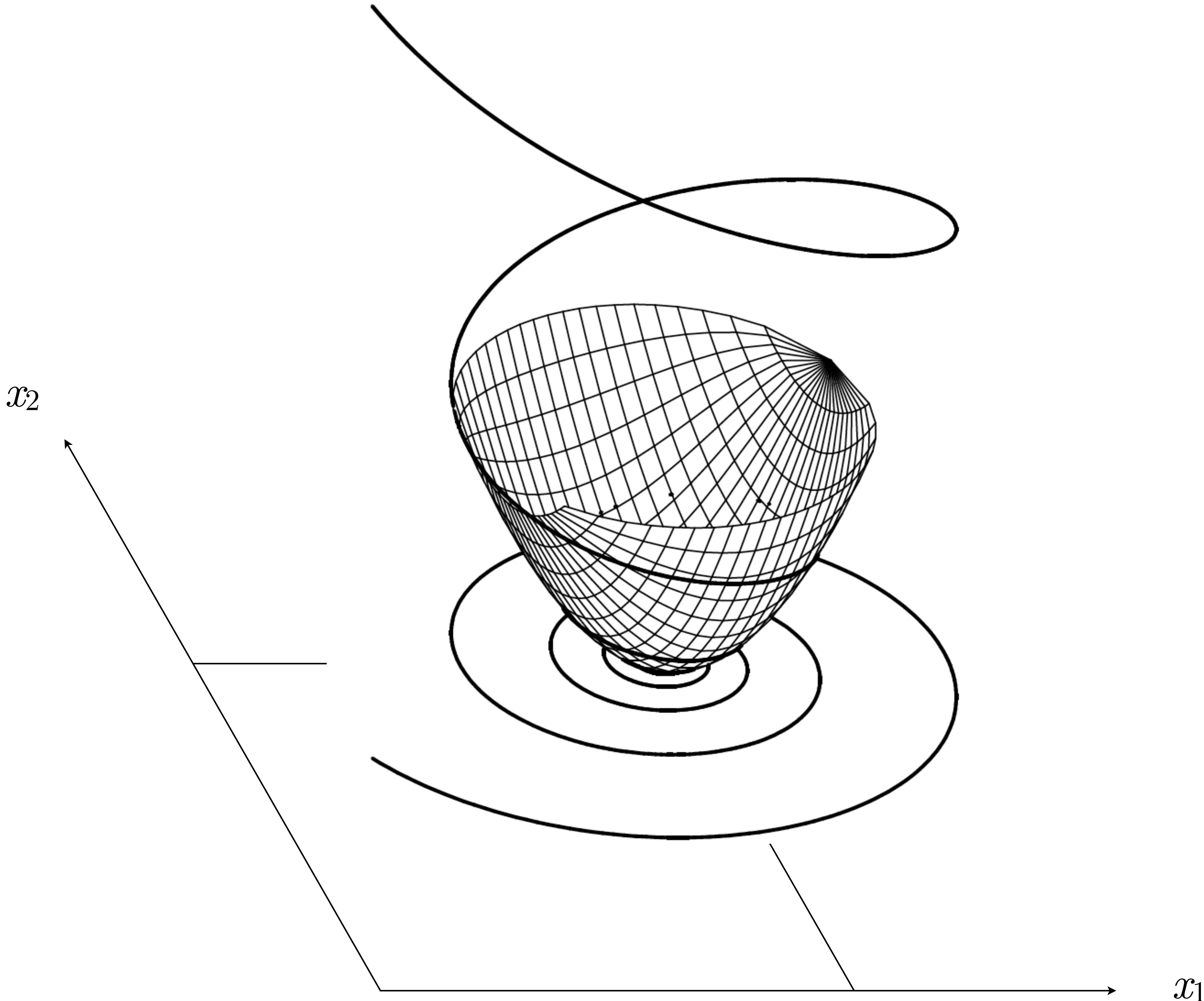
Theorem 1. *Let V be a non-negative function on $\mathbb{R}^n$ and let $\dot{V}$ represent the time derivative of V along trajectories of the system dynamics* (1)*:*

$$\dot{V} = \frac{\partial V}{\partial x}\frac{dx}{dt} = \frac{\partial V}{\partial x}F(x).$$

Let $B_r = B_r(0)$ be a ball of radius r around the origin. If there exists $r > 0$ such that V is positive definite and $\dot{V}$ is negative semi-definite for all $x \in B_r$, then $x = 0$ is stable in the sense of Lyapunov. If V is positive definite and $\dot{V}$ is negative definite in B_r, then $x = 0$ is locally asymptotically stable.



A general task about the stability of motion
(Общая задача об устойчивости движения)


x_2
x_1

How to find a Lyapunov function?

For linear systems:

- Let $\dot{x} = Ax.$
- V positive definite ⇔ P positive definite matrix (P > 0): $V(x) = x^\top Px$
- Compute its derivative: $x^\top Px > 0 \text{ for all } x \neq 0$

$$\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} = x^T(A^TP + PA)x \qquad Q = A^TP + PA < 0$$

- Solve for P and verify whether $P > 0$

For every negative definite $Q = Q^{\mathsf{T}}$ there exits a positive definite $P = P^{\mathsf{T}}$ such that

$$Q = A^T P + PA < 0$$

If and only if

$$Re\{\lambda_i\} < 0$$

for all i

Real part of the eigenvalues of A are negative determines the stability properties of linear systems

Today

- Stability of a subclass of hybrid systems
- Stability properties of switched systems
  - Multiple Lyapunov functions
  - Common Lyapunov function
  - ➡ Do solutions converge to an equilibrium?
  - ➡ How to choose a stabilizing sequence?
- Next class: how to extend stability results to a larger class of hybrid systems

Equilibrium point

Definition 4.1 (Equilibrium) *$x = 0 \in \mathbb{R}^n$ is an equilibrium point of H if:*

1. *$f(q, 0) = 0$ for all $q \in \mathbf{Q}$, and*
2. *$((q, q') \in E) \wedge (0 \in G(q, q')) \Rightarrow R(q, q', 0) = \{0\}$.*

Proposition 4.2 *If $(q_0, 0) \in \mathrm{Init}$ and $(\tau, q, x) \in \mathcal{H}_{(q_0,0)}$ then $x(t) = 0$ for all $t \in \tau$.*

- If the continuous part of the state starts on the equilibrium point, it stays there forever.
- One would like to characterize the notion that if the continuous state starts close to the equilibrium point it stays close, or even converges to it.
- Use the definitions of Lyapunov for this purpose.

Stability concepts

Definition 4.3 (Stable Equilibrium) *Let $x = 0 \in \mathbb{R}^n$ be an equilibrium point of H. $x = 0$ is stable if for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $(\tau, q, x) \in \mathcal{H}_{(q_0, x_0)}$ with $\|x_0\| < \delta$, $\|x(t)\| < \epsilon$ for all $t \in \tau$.*

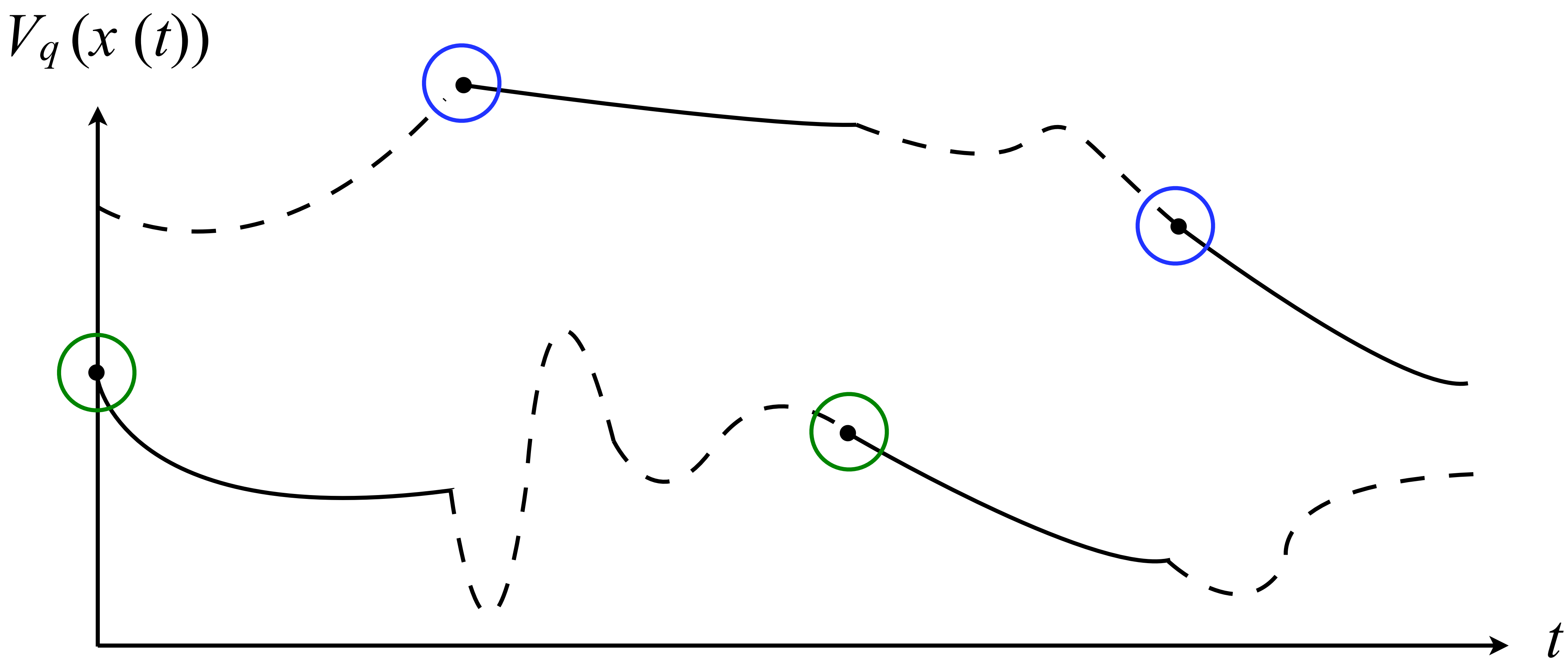
Definition 4.4 (Asymptotically Stable Equilibrium) *Let $x = 0 \in \mathbb{R}^n$ be an equilibrium point of H. $x = 0$ is asymptotically stable if it is stable and there exists $\delta > 0$ such that for all $(\tau, q, x) \in \mathcal{H}^{\infty}_{(q_0, x_0)}$ with $\|x_0\| < \delta$,* $\lim_{t \to \tau_\infty} x(t) = 0$ *where* $\tau_\infty = \sum_i (\tau_i' - \tau_i)$.

Stability analysis

Theorem 4.9 *Consider a hybrid automaton H with $x = 0$ as an equilibrium point, $|\mathbf{Q}| < \infty$, and $R(q, q', x) = \{x\}$. Consider an open set $D \subseteq \mathbf{X}$ with $0 \in D$ and a function $V : \mathbf{Q} \times D \to \mathbb{R}$ continuously differentiable in x such that for all $q \in \mathbf{Q}$:*

1. $V(q, 0) = 0$,
2. $V(q, x) > 0$ *for all* $x \in D \setminus \{0\}$,
3. $\frac{\partial V}{\partial x}(q, x) f(q, x) \leq 0$ *for all* $x \in D$.

If for all $(\tau, q, x) \in \mathcal{H}$ and all $\hat{q} \in \mathbf{Q}$ the sequence $\{V(q(\tau_i), x(\tau_i)) \; : \; q(\tau_i) = \hat{q}\}$ is non increasing, then $x = 0$ is a stable equilibrium of H.


$V_q(x(t))$
t

Stability analysis

Corollary 4.11 *Consider a hybrid automaton H with $x = 0$ an equilibrium point, $|\mathbf{Q}| < \infty$, and $R(q, q', x) = \{x\}$. Consider an open set $D \subseteq \mathbf{X}$ with $0 \in D$ and assume there exists a function $V : D \to \mathbb{R}$ continuously differentiable in x such that:*

1. $V(0) = 0$,
2. $V(x) > 0$ *for all* $x \in D \setminus \{0\}$,
3. $\frac{\partial V}{\partial x}(x) f(q, x) \leq 0$ *for all* $q \in \mathbf{Q}$ *and all* $x \in D$.

Then $x = 0$ is a stable equilibrium of H

Proof: Define $\hat{V} : \mathbf{Q} \times \mathbf{X} \to \mathbb{R}$ by $\hat{V}(q, x) = V(x)$ for all $q \in \mathbf{Q}$, $x \in \mathbf{X}$ and apply Theorem 1. $\blacksquare$

Stability analysis

Corollary 4.12 *Consider a hybrid automaton H with $x = 0$ an equilibrium point, $|\mathbf{Q}| < \infty$, and assume $R(q, q', x)$ is non-expanding. Consider an open set $D \subseteq \mathbf{X}$ with $0 \in D$ and a function $V : \mathbf{Q} \times D \to \mathbb{R}$ continuously differentiable in x such that for all $q \in \mathbf{Q}$:*

1. $V(q, 0) = 0$,
2. $V(q, x) > 0$ *for all* $x \in D \setminus \{0\}$,
3. $\frac{\partial V}{\partial x}(q, x) f(q, x) \leq 0$ *for all* $x \in D$.

If for all $(\tau, q, x) \in \mathcal{H}$ and all $\hat{q} \in \mathbf{Q}$ the sequence $\{V(q(\tau_i), x(\tau_i)) \ : \ q(\tau_i) = \hat{q}\}$ is non increasing, then $x = 0$ is a stable equilibrium of H

Theorem 4.13 *Consider a hybrid automaton H with $|\mathbf{Q}| < \infty$. Consider an open set $D \subseteq \mathbf{X}$ with $0 \in D$ and a function $V : \mathbf{Q} \times D \to \mathbb{R}$ continuous in x with $V(q, 0) = 0$ and $V(q, x) > 0$ for all $x \in D \setminus \{0\}$. Assume that for all $(\tau, q, x) \in \mathcal{H}$ the sequence $\{V(q(\tau_i), x(\tau_i))\}$ is non increasing and that there exists a continuous function $g : \mathbb{R}^+ \to \mathbb{R}^+$ with $g(0) = 0$, such that for all $t \in [\tau_i, \tau_i']$, $V(q(t), x(t)) \leq g((V(q(\tau_i), x(\tau_i))$. Then $x = 0$ is a stable equilibrium of H*