Objective

1. To analyze the dynamics of a bicycle with rear wheel steering.
2. To compare the performance of two control schemes, a feedback controller and a PID controller, while applied to the same system.

Problem 1 - Rear steer bicycle

Analyzing it as a dynamic system, the controllability properties of a moving bicycle depend on the feedback mechanism created by its design, especially the position of its steering wheel. Obtaining a detailed model for the bicycle dynamics is often an elaborate task because the system has many degrees of freedom. The objective of this this exercise is to find an approximate model based on simplified dynamics.

"To derive the equations of motion we assume that the bicycle rolls on the horizontal xy plane. Introduce a coordinate system that is fixed to the bicycle with the ξ through the contact points of the wheels with the ground, the η -axis horizontal and the ζ -axis vertical, as shown in Figure 1. Let v_0 be the velocity of the bicycle at the rear wheel, b the wheel base, φ the tilt angle and δ the steering angle. The coordinate system rotates around the point O with the angular velocity $\omega = v_0\ \delta/b$, and an observer fixed to the bicycle experiences forces due to the motion of the coordinate system.

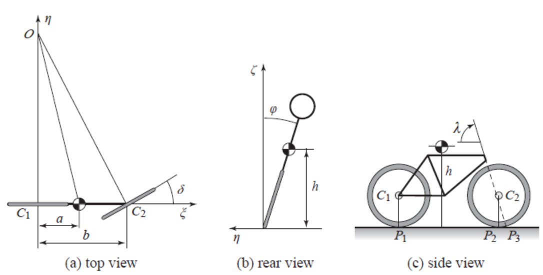

Figure 1: Schematic views of a bicycle

Let m be the total mass of the system, J the moment of inertia of this body with respect to the ξ -axis and $D = mah$ the product of inertia with respect to the ξ ζaxes. Furthermore, let the ξ and ζcoordinates of the center of mass with respect to the rear wheel contact point, P_1, be a and h, respectively. We have $J = mh^2$ and D = mah. The torques acting on the system are due to gravity and centripetal action. Assuming that the steering angle δ is small,the equation of motion becomes:

$$J\frac{d^2\varphi}{dt^2} - \frac{Dv_0}{b}\frac{d\delta}{dt} = mghsin(\varphi) + \frac{m{v_0}^2h}{b}\delta \tag{1}$$

The term $mghsin(\varphi)$ is the torque generated by gravity. The terms containing δ and its derivative are the torques generated by steering, with the term $\frac{Dv_0}{b}\frac{d\delta}{dt}$ due to inertial forces and the term $\frac{m{v_0}^2h}{b}\delta$ due to centripetal forces." [1]

[1] Taken from http://www.cds.caltech.edu/~murray/books/AM08/pdf/fbs-examples_25Sep16.pdf

The design of the structural system has a big impact in the analysis of the bicycle dynamics, because of that, the torque and the inclination are defined and applied to the handlebar and the frame in the control variable.

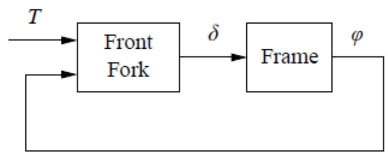

Figure 2: Block diagram of the bicycle with a front wheel steering

Under certain conditions, the feedback can actually stabilize the bicycle. A crude empirical model is obtained by assuming that the feedback control law performed by the fork can be modelled as the static system

$$\delta = k_1(V)T - k_2(V)\varphi \tag{2}$$

Procedure

- [0.5 points] Develop a model of the bicycle based on the previous information and show the procedure. Make a detailed description of the aspects and assumptions used for the development of the model. You must measure the values of a, b, h and m.
- [0.5 points] Find the equilibrium points of your model and show the procedure.
- [0.5 points] For each equilibrium point, use the Lyapunov theorem to determine its stability properties. Why is it important to prove that the equilibrium points are stable?
- [0.5 points] If there are unstable equilibrium points, how can you stabilize the system? Make a quantitative and qualitative analysis.
- [0.5 points] Is your system reachable? Justify your answer.
- [1.0 point] Implement a feedback controller for the system based on your model. Verify your results using simulations.
- [1.2 points] Implement a PID controller for the system based on your model. The stable state error must be equal to zero (if possible) and the response time must be 5% faster than that of the closed loop system. Verify your results using simulations.
- [0.3 points] For each controller, find:
  (a) Peak time
  (b) Maximum overshoot
  (c) Settling time

  Show the procedure.

General Remarks:

Remember that the report must include:

- A mathematical model (state space).
- A detailed procedure.
- The answer to all the above questions.
- Conclusions. Include a comparison between the experience of driving the bicycle and the analytic results.
- References.

Finally, please send the teaching assistant an email with the *Mathematica* Notebook that you used.

References

[1] K. Åström and R. Murray, *Feedback Systems: An Introduction for Scientists and Engineers*, 2nd ed, Princeton University Press, 2016.

[2] K. Åström, R. Klein and Anders Lennartsson, *Bicycle Dynamics and Control: Adapted bicycles for education and research*, Control Systems Magazine, IEEE, 2005.