Objective

1. To learn how to use *Mathematica* to run systems analysis.
2. To understand and verify the concept of stability and its properties in a dynamic system.

Pre-lab - Cart-pendulum system

The cart with an inverted pendulum, shown below, is pushed to the right with an impulse force F. It is assumed that the motion takes place in a vertical plane, see Fig. 1.

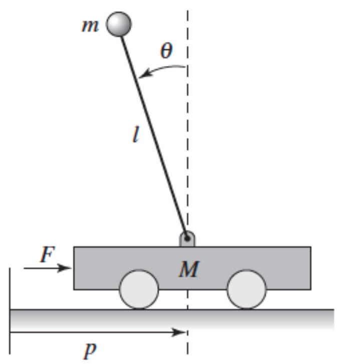

Figure 1: Cart-pendulum. [1]

If we are interested only in analyzing the pendulum's upright orientation, without worrying about the location of base of the cart, we can reduce the system to the one shown in Fig. 2.

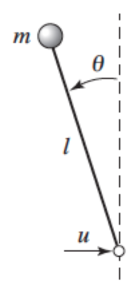

Figure 2: Inverted pendulum. [1]

The dynamics of this simplified system are given by

$$(I + ml^2)\ddot{\theta} - mglsin\theta = lucos\theta - \gamma\dot{\theta}$$

where m is the mass of the pendulum, γ is the friction of the cart, l represents the length of the pendulum, I corresponds to the inertia of the pendulum, u represents the force applied at the base, and θ is the pendulum angle from vertical.

Procedure

Consider the following parameters:

- $m = 0.2$ kg
- $\gamma = 0.6 N/m/sec$
- $l = 0.3$ m
- $I = 0.6\ kg * m^2$

According to the model suggested before:

- Derive the input/state/output equations; i.e., write the equations in the form

$$\frac{dx}{dt} = f(x; u), y = h(x)$$

- Find all equilibrium solutions. Explain what does each point represent in the physical system.
- Characterize the stability properties of each equilibrium point and plot the phase-portrait of the system. Show the procedure.

Also, an inverted pendulum can be described by the following normalized model

$$\frac{dx_1}{dt} = x_2, \quad \frac{dx_2}{dt} = sinx_1 + ucosx_1$$

- Using the feedback law $u = -2sinx_1 - x_2cosx_1$, show that the origin (0,0) is a locally asymptotically stable equilibrium point.

Problem 1 - Predator-Prey (Lotka-Volterra)

Consider a pair of species, one of which consists of predators whose population is denoted by y, the other species is its prey, which has a population denoted by x. The *Lotka-Volterra* system which describes their behavior is

$$\dot{x} = xa - bxy$$
$$\dot{y} = -cy + dxy$$

where $a = 10$, $b = 2$, $c = 30$ and $d = 3$.

Procedure

Based on the *Lotka-Volterra* model for the two species:

- [1 point] Using simulations, explain the evolution of the populations over time.
- [1 point] Find all equilibrium solutions and characterize the stability properties of each equilibrium point. Show the procedure. Plot the phase-diagram of the system and explain the result.

Problem 2

Consider a system described by

$$\frac{dx_1}{dt} = x_2 + x_1(1 - x_1^2 - x_2^2), \quad \frac{dx_2}{dt} = -x_1 + x_2(1 - x_1^2 - x_2^2)$$

Procedure

Based on the given system:

- [1 point] Characterize the stability properties of the equilibrium points, show the procedure and verify using the phase-diagram of the system.
- [0.5 points] Illustrate the Lyapunov's concept of a stable solution using different initial conditions and showing the evolution of the states around the equilibrium.

Problem 3

Consider a system of the form

$$\dot{x}(t) = Ax(t)$$

where

$$\dot{x_1} = -x_1 + 4x_2$$

$$\dot{x_2} = -x_1 - x_2^3$$

and the Lyapunov function

$$V = x_1^2 + ax_2^2$$

Procedure

Based on the given system and using the previous Lyapunov function:

- [0.5 points] Find how many equilibrium points does the system have. Show the procedure.
- [1 point] Using the Lyapunov stability theorem, find the value of a for which you can state that the origin is globally asymptotically stable. Verify your result with a simulation.

General Remarks:

Remember that the report must include:

- A mathematical model (state space).
- A procedure.
- The answer to all the above questions.
- Conclusions.
- References.

Finally, please send the teaching assistant an email with the *Mathematica* Notebook that you used.

References

[1] K. Åström and R. Murray, *Feedback Systems: An Introduction for Scientists and Engineers*, 2nd ed, Princeton University Press, 2016.