2016 American Control Conference (ACC)
Boston Marriott Copley Place
July 6-8, 2016. Boston, MA, USA

Stability Properties of Reciprocal Networks

Isabel Fernández[1] and Jorge Finke[2]

Abstract— **One of the aims of network formation models is to explain salient properties of empirical networks based on simple mechanisms for establishing links. Such mechanisms include random attachment (a generic abstraction of how new incoming nodes connect to a network), triadic closure (how the new nodes establish transitive relationships), and network response (how nodes react to new attachments). Our work analyzes the combined effect of the three mechanisms on various local and global network properties. In particular, we derive an expression for the asymptotic behavior of the local reciprocity coefficient as a function of the in-degree of a node. Furthermore, we show that the dynamics of the global reciprocity and the global clustering coefficients correspond to time-varying linear systems. Finally, we identify conditions under which the equilibria of both coefficients are asymptotically stable.**

I. Introduction

Reciprocal relationships, established by the practice of giving one thing and receiving back another, lie at the heart of numerous interconnected systems in which users enjoy mutual benefits (e.g., trust) based on a collaborative exchange of information. Reciprocity, in its simplest form, represents a potential feedback loop, where a target node responds to an action, an event, or a process of a source node. Networks with a high number of such loops define a class of directed networks called reciprocal networks [1].

Recent research in the field of network theory has focused on developing models that explain the structure of empirical networks, including the formation of patterns found in measures of degree centralities (in- and out-degree distributions), clustering (transitive relationships), community structures (modularity), and assortative mixing (homophily) [2]–[4]. Understanding these properties represents a first step toward designing estimation and control algorithms that take account of emerging structures inherent to a class of networks. The work in [4] illustrates, for example, how undirected networks rely on connectivity patterns for their function, in particular, how resilience levels depend on the underlying degree distribution and clustering properties. Less attention has been paid to the development of analytical frameworks that help assess such relationships for reciprocal networks.

To help bridge this gap, this paper explores the coupled dynamics of reciprocity and clustering in networks with power law degree distributions. Our work is closely related to the growth models introduced in [5] and [6], where the authors propose different frameworks to evaluate how reciprocal edges impact the equilibrium distributions of the degree of nodes. Simple mechanisms for establishing links explain the emergence of empirical power law distributions (e.g., in networks where nodes represent online articles and hyperlinks connect one article with another). As in [5], [6], we assume that the probability of establishing reciprocal edges does not depend on the degree of nodes or any other network measure; however, we focus on identifying relationships in the outcome resulting from mechanisms of triadic closure and reciprocity.

Conceptually, our work is also related to [7], where the authors formalize a notion of robustness against link perturbations. Their work introduces a metric on the space of all weighted networks and shows that certain centrality measures satisfy the condition of Lipschitz continuity, meaning that any perturbation in the weight of a link leads to a difference in centrality that is bounded. However, without developing a formal time- or event-driven model, it is not possible to evaluate whether small perturbations remain close to an equilibrium state. In other words, whether a given centrality measure is indeed stable (in the sense of Lypunov). Providing such a framework remains in general an open challenge.

The main motivation behind our work is to study the stability of equilibrium patterns in clustering and reciprocity. We use the event-driven model in [8] that combines three simple mechanisms to generate reciprocal networks with varying degrees of clustering: *Random attachment* describes how a new node connects to a given network, *triadic closure* establishes transitivity relationships, and *network response* characterizes the way nodes react to new attachments. Our aim is to identify conditions under which the dynamics resulting from the interaction of these mechanisms can be approximated as a set of difference equations. In particular, we derive closed-loop expressions for the global reciprocity and the global clustering coefficients, and show that both coefficients are asymptotically stable.

II. Preliminaries

Consider a ordered set of graphs $\mathcal{G} = \{\mathcal{G}(0), \mathcal{G}(1), \ldots\}$. Each element $\mathcal{G}(t) = (\mathcal{H}(t), \mathcal{A}(t))$ represents a network with a set of nodes $\mathcal{H}(t) = \{1, \ldots, N_t\}$ and a set of directed edges $\mathcal{A}(t) = \{(i,j) : i,j \in \mathcal{H}(t)\}$. The pair $(j,i) \in \mathcal{A}(t)$ indicates an edge from node j to node i, and $k_i(t)$ and $\hat{k}_i(t)$ represent the in- and out-degree of node $i \in \mathcal{H}(t)$ at time t. The occurrence of an event at time t represents nodes establishing new edges based on the following mechanisms.

M1 *Random attachment:* A new incoming node links to $m \geq 1$, $m \in \mathbb{Z}^+$, different nodes, selected according to a uniform random distribution over $\mathcal{H}(t-1)$.

[1]Department of Electrical and Computer Engineering, The Ohio State University, Columbus, Ohio.
[2]Department of Electrical Engineering and Computer Science, Pontificia Universidad Javeriana, Cali, Colombia.

M2 *Triadic closure:* For every edge that a new node establishes during random attachment, it tries to establish an additional edge within the new neighborhood. In particular, if node $j \notin \mathcal{H}(t-1)$ connects to some node $j' \in \mathcal{H}(t-1)$, it connects to an outgoing neighbor of node j' with probability $0 < \pi_f \leq 1$.

M3 *Network response:* There are two ways nodes respond to the attachment of a new node. The first approach is based on reciprocity: Each of the m randomly selected nodes establishes an edge to the new node with probability $0 \leq \pi_r \leq 1$. The second approach shows no preference for establishing reciprocal edges: A set of $n \geq 0$ randomly selected nodes connect to the new node.

Let $\in \mathcal{T} = \{0, 1, 2, \ldots\}$ be a set of time indices. Consider the following assumptions.

A1 *The initial network*: The network $\mathcal{G}(0)$ is weakly connected and has more than $2m$ nodes, each with at least one outgoing neighbor.

A2 *The event-time incidence rate*: Mechanisms M1-M3 are triggered every index $t \in \mathcal{T}$.

Under assumptions A1-A2, it can be shown that as $t \to \infty$ the in- and out-degree distributions obey an extended power law and an exponential law (see Theorems 1 and 2 in [8], respectively). Here we focus on characterizing the reciprocity and clustering properties of $\mathcal{G}(t)$. Consider the following definitions.

Definition 1 (path of length two): A *path of length two* is an ordered sequence $p_{ij}(t) = (x_0 = i, x_1, x_2 = j)$ such that $(x_i, x_{i+1}) \in \mathcal{A}(t)$ for $i = 0, 1$.

In other words, a path p_{ij} of length two is a sequence of three nodes that connects node i to node j.

Definition 2 (triad): Consider three nodes $i, j, j' \in \mathcal{H}(t)$ such that $(j', j) \in \mathcal{A}(t)$. A subgraph is said to be a *triadic closure* or *triad* involving node j' if

a) $(j', i) \in \mathcal{A}(t)$ and either (j, i) or $(i, j) \in \mathcal{A}(t)$; or
b) $(i, j') \in \mathcal{A}(t)$ and $(i, j) \in \mathcal{A}(t)$.

According to Definition 2, a triad represents any possible configuration of three edges between three nodes, except closed cycles ($\{(j', i), p_{j'i}\}, \{(j', j), p_{j'j}\}$, and $\{(i, j), p_{ij}\}$ represent triads). Note also that because a triad is the combination of a direct link and a path of length two, it represents two alternative ways for a source node to reach a target node. That is, a triad does not include an edge between the target node and the source node (e.g., how a target node may respond to a process of a source node).

Definition 3 (reciprocal cycle): A *reciprocal cycle* involving node i is a path of length two that starts and ends at node i, that is, $p_{ii}(t) = (i, j, i)$ for some $j \in \mathcal{H}(t)$.

Note that a reciprocal cycle groups a link and its reciprocal segment and represents the lowest-order cycle in a directed network, i.e., the simplest path configuration except for self-loops. Edges that are part of reciprocal cycles capture either a stimulus of a source node or a response by a target node, and are called reciprocal edges.

III. Local and Global Reciprocity Coefficients

The reciprocity coefficient of a node describes the ratio between the number of reciprocal edges involving that node and its total degree. Let $r(k)$ represent the average local reciprocity coefficient of nodes with in-degree k. The following theorem presents sufficient conditions for the convergence of the local reciprocity coefficient.

Theorem 1 (local reciprocity): Suppose that assumptions A1-A2 hold. Moreover, suppose the reciprocal approach for mechanism M3 satisfies $\pi_r > 0$. The asymptotic behavior of the local reciprocity coefficient for nodes with in-degree k approaches

$$r^*(k) = \frac{2m\pi_r\,(1 + g(k))}{k + m(\pi_f + 1) + (n + m\pi_r)g(k)}$$

where

$$g(k) = (\alpha - 1)\ln\left(\frac{k\pi_f + m(1 + \pi_f + \pi_r) + n}{(\pi_f + 1)(m(1 + \pi_r) + n)}\right)$$

and $\alpha = 2 + \frac{1}{\pi_f} + \frac{\pi_r}{\pi_f} + \frac{n}{m\pi_f}$.

Proof: To capture the average local reciprocity coefficient of nodes with in-degree k, consider node i (selected uniformly at random) with in-degree $k_i(t) = k$. Let $r_i(t)$ represent the number of reciprocal edges involving node i at time t. First, we will characterize the rate of change of $r_i(t)$. According to mechanisms M1 and M3, and assumption A2, at time $t = t_j \neq t_i$, node $j \notin \mathcal{H}(t-1)$ connects to m different nodes; and each of these nodes may establish a reciprocal connection to node j. The probability that node j attaches to node $i \in \mathcal{H}(t-1)$ at time t, and that node i establishes a reciprocal edge is given by $\frac{2m\pi_r}{N_{t-1}}$. The overall rate of change of $r_i(t)$ is given by

$$\frac{dr_i(t)}{dt} = \frac{2m\pi_r}{N_{t-1}} \tag{1}$$

with boundary condition $r_i(t_i) = 2m\pi_r$, which corresponds to the expected number of reciprocal edges established at time t_i (i.e., when node i joined the network). The solution to eq. (1) is

$$r_i(t) = 2m\pi_r\left(1 + \ln\left(\frac{N_{t-1}}{N_{t_i-1}}\right)\right) \tag{2}$$

Moreover, it can be shown that the in-degree of node i at time t equals

$$k_i(t) = (m(1+\pi_r)+n)\left(\frac{N_{t-1}}{N_{t_i-1}}\right)^{\frac{1}{\alpha-1}}\left(1 + \frac{1}{\pi_f}\right) - (\alpha-1)m$$

where $\alpha = \left(2 + \frac{1}{\pi_f} + \frac{\pi_r}{\pi_f} + \frac{n}{m\pi_f}\right)$ (see the proof of Theorem 1 in [8] for details). Rearranging the above equation, we get

$$\frac{N_{t-1}}{N_{t_i-1}} = \left(\frac{k_i(t)\pi_f + m(1 + \pi_f + \pi_r) + n}{(\pi_f + 1)(m(1 + \pi_r) + n)}\right)^{\alpha-1} \tag{3}$$

Replacing eq. (3) in eq. (2), the number of reciprocal edges involving node i in terms of the in-degree of node i is given

by

$$r_i(t) = 2m\pi_r\left(1 + g(k_i(t))\right) \tag{4}$$

where

$$g(k_i(t)) = (\alpha - 1)\ln\left(\frac{k_i(t)\pi_f + m(1+\pi_f+\pi_r) + n}{(\pi_f+1)(m(1+\pi_r)+n)}\right)$$

Moreover, using Lemma 1 in [8], if t is sufficiently large, then the out-degree of node i can be described in terms of its in-degree as

$$\hat{k}_i(t) = m(1+\pi_f) + (n + m\pi_r)g(k_i(t))$$

For a fixed time index t, the total degree of node i is $k_i + \ m(1+\pi_f) + (n+m\pi_r)g(k_i)$. Dividing the expected number of reciprocal edges (eq. (4)) over the total degree of node i, we know that the asymptotic behavior of the average local reciprocal coefficient for nodes with in-degree $k_i = k$ satisfies

$$r^*(k) = \frac{2m\pi_r(1+g(k))}{k + m(1+\pi_f) + (n+m\pi_r)g(k)} \tag{5}$$

■

Theorem 1 implies that the local reciprocity coefficients of the networks in $\mathcal{G}$ do neither vanish nor depend on the initial network $\mathcal{G}(0)$, which also holds for the *global* reciprocity coefficient. The global reciprocity coefficient is defined as the ratio between the number of reciprocal edges and the total number of edges across the entire network. The following theorem characterizes its asymptotic behavior.

Theorem 2 (global reciprocity): Suppose that assumptions A1-A2 hold. Moreover, suppose the reciprocal approach for mechanism M3 satisfies $\pi_r > 0$. The global reciprocity coefficient $r_g(t)$ converges to

$$R^* = \frac{2m\pi_r}{m(1+\pi_f+\pi_r)+n}$$

Proof: We know that when a node attaches to the network, the number of reciprocal edges that are formed is $2m\pi_r$. Moreover, the combination of the random approach for mechanism M3 and mechanisms M1 or M2 increases the expected number of reciprocal edges at time t by $\frac{2mn}{N_{t-1}} + \frac{2m\pi_f n}{N_{t-1}}$. So, the total number of reciprocal edges across the network increases every time instant by

$$2m\left(\pi_r + \frac{n(1+\pi_f)}{N_{t-1}}\right) \tag{6}$$

Let L_0^r represent the number of reciprocal edges and L_0 the total number of edges of the initial network. The average number of reciprocal edges at time t is

$$L_0^r + 2m\pi_r t + 2mn(1+\pi_f)\sum_{\tau=1}^{t-1}\frac{1}{N_\tau} \tag{7}$$

Moreover, when a node attaches to the network, mechanisms M1-M3 establish on average $m+m\pi_f+m\pi_r+n$ new edges. The average number of total edges in the network (including reciprocal edges) at time t is

$$L_0 + (m(1+\pi_f+\pi_r)+n)t \tag{8}$$

The global reciprocity coefficient of the network is given by the ratio of eq. (7) to eq. (8), which equals

$$R(t) = \frac{L_0^r + 2m\pi_r t + 2mn(1+\pi_f)\sum_{\tau=1}^{t-1}\frac{1}{N_\tau}}{L_0 + (m(1+\pi_f+\pi_r)+n)t} \tag{9}$$

For sufficiently large values of t, $L_0 \ll (m(1+\pi_f+\pi_r)+n)t$. Moreover, due to assumption A2, if a new node attaches to the network every time index t, the number of nodes at time $t-1$ is $N_{t-1} = N_0 + t - 1$ where N_0 indicates the number of nodes of the initial network. As $t \to \infty$ the first and third term in the numerator of eq. (9) tend to zero and $R(t)$ converges to

$$R^* = \frac{2m\pi_r}{m(1+\pi_f+\pi_r)+n} \tag{10}$$

■

Theorem 2 shows that the asymptotic behavior of the global reciprocity coefficient does not depend of the initial conditions of the network.

IV. Global clustering coefficient

To further investigate the opposite effects of triad formation and the reciprocal approach consider the following measure of triadic closure. The global clustering coefficient is defined as the ratio between the total number of triads across the entire network and the the total number of paths of length two that could potentially lead to a triad. The following theorem describes the asymptotic behavior of the global clustering coefficient.

Theorem 3 (global clustering): Suppose that assumptions A1-A2 hold. The global clustering coefficient $C(t)$ converges to

$$C^* = \frac{m\pi_f(1+\pi_r)}{(m(1+\pi_f+\pi_r)+n)^2 + m(1+\pi_f)(n+m\pi_r)}$$

Proof: Note that when a node attaches to the network (based on mechanism M1), it forms, on average, $m\pi_f$ triads due to mechanism M2, and $m\pi_f\pi_r$ triads due to the reciprocal approach for mechanism M3, for a total of $m\pi_f + m\pi_f\pi_r$ every time t. Moreover, let

$$a_1 = m(1+\pi_f+\pi_r)+n \tag{11}$$

denote the expected total number of edges (incoming and outgoing) formed when a new node attaches to the network. Note that m edges are formed according to the random attachment process, $m\pi_f$ edges according to triadic closure, and $m\pi_r+n$ edges according to both approaches for network response. Note also that additional triads may be formed due to edges established in the following ways by the mechanisms. The probability of forming a triad with edges established by mechanism M1, combined with the reciprocal approach for mechanism M3, is $\frac{1}{N_{t-1}}m(m-1)a_1(2+\pi_r)$. Similarly, the probability of forming a triad with two edges established by mechanism M2 is $\frac{1}{N_{t-1}}m\pi_f((m-1)\pi_f)2a_1$. The probability that an edge established by the random approach for mechanism M3 combined an edge established by one of the mechanisms M1-M3 (i.e., random attachment,

triadic closure, or the random approach for M3) increases the number of triads by

$$\frac{1}{N_{t-1}}nma_1(1+\pi_r)+\frac{1}{N_{t-1}}nm\pi_f a_1+\frac{1}{N_{t-1}}n(n-1)2a_1$$

On average the number of triads formed when a new node attaches to the network increases by

$$m\pi_f(1+\pi_r)+\frac{1}{N_{t-1}}a_0a_1 \tag{12}$$

where $a_0 = m(m-1)(2(1+\pi_f)+\pi_r)+n(2(n-1)+m(1+\pi_f+\pi_r))$. The average number of triads at time t is

$$F_0+m\pi_f(1+\pi_r)t+a_0a_1\sum_{\tau=1}^{t-1}\frac{1}{N_\tau} \tag{13}$$

where F_0 is the number of triads of the initial network.

To estimate the total number of paths of length two that are formed when node i attaches to the network, consider the following three cases. First, consider a path $p_{ij} = (i, j'j)$, i.e., when a path starts at node i. Node i has on average $m + m\pi_f$ outgoing edges established through events triggered by mechanisms M1 and M2. Each outgoing neighbor of node i has on average $m + m\pi_f + m\pi_r + n$ outgoing neighbors, so that the number of paths of length two involving node i increases by $m(1 + \pi_f)a_1$. Second, consider a path $p_{ji} = (j, j', i)$ that ends with an incoming edge at node i. Node i has $n + m\pi_r$ incoming neighbors (established based on mechanism M3), and each incoming neighbor has on average $m+m\pi_f+m\pi_r+n$ incoming edges. The number of paths of length two involving node i increases every time instant by $(n + m\pi_r)a_1$. Finally, consider a path $p_{jj'} = (j, i, j')$ where node i lies in the middle. Because node i has $m+m\pi_f$ outgoing edges and $n+m\pi_r$ incoming edges, the number of paths of length two increases every time t by $m(1 + \pi_f)(n + m\pi_r)$. So the average number of paths of length two that a new node establishes when it attaches to the network is

$$a_1^2+m(1+\pi_f)(n+m\pi_r) \tag{14}$$

Multiplying eq. (14) by the number of nodes in the network and taking into account the initial network, the total number of paths at time t yields

$$P_0+(a_1^2+m(1+\pi_f)(n+m\pi_r))t \tag{15}$$

where P_0 is the number of paths of length two of the initial network. The global clustering coefficient is given by the ratio between eq. (13) and eq. (15), which equals

$$C(t)=\frac{F_0+m\pi_f(1+\pi_r)t+a_0a_1\sum_{\tau=1}^{t-1}\frac{1}{N_\tau}}{P_0+(a_1^2+m(1+\pi_f)(n+m\pi_r))\,t} \tag{16}$$

Since $a_1 = m(1 + \pi_f + \pi_r) + n$, if t is sufficiently large, then $P_0 \ll (a_1^2 + m(1 + \pi_f)(n + m\pi_r))t$. Moreover, due to assumption A2, starting from an initial network with N_0 nodes, the number of nodes at time $t-1$ is $N_{t-1} = N_0+t-1$. As $t \to \infty$, the global clustering coefficient $C(t)$ converges to

$$C^*=\frac{m\pi_f(1+\pi_r)}{(m(1+\pi_f+\pi_r)+n)^2+m(1+\pi_f)(n+m\pi_r)} \tag{17}$$

■

Theorem 3 implies that the asymptotic behavior of $C(t)$ does not depend on the initial network $\mathcal{G}(0)$. In general, the resulting global clustering coefficient ranges between $0 \le C^* \le 0.25$. Highly-clustered networks are formed when reciprocity is low and every new node attaches to only one node ($m = 1$).

V. Dynamics of the global coefficients

Next, we derive the closed-loop forms of both the global reciprocity and the global clustering coefficients. Using eq. (11), note that the total number of edges of $\mathcal{G}(t)$ is L_0+a_1t. Moreover, using eq. (6) we also know the expected number of reciprocal edges that are formed when a new node attaches to the network at time t. The total number of reciprocal edges at $t+1$ as a function of $R(t)$ is

$$R(t)(L_0+a_1t)+2\left(m\pi_r+\frac{mn}{N_t}+\frac{m\pi_f n}{N_t}\right) \tag{18}$$

The first term in eq. (18) represents the number of reciprocal edges at time t, and the second term denotes the number of new reciprocal edges at time $t+1$ (i.e., the expected number of reciprocal edges formed when a new node attaches to the network). In particular, note that as $t \to \infty$ the number of new reciprocal edges converges to $2m\pi_r$. However, based on assumption A2, $N_t = N_0 + t$, and for small values t, the combination of edges established by mechanism M3 and by mechanisms M1 or M3 will lead to additional reciprocal edges $\left(\frac{mn}{N_t}\text{ or }\frac{m\pi_f n}{N_t}\text{, respectively}\right)$. The global reciprocity coefficient of the network at $t+1$ is given by

$$R(t+1)=\frac{(L_0+a_1t)R(t)+2m\left(\pi_r+\frac{n(1+\pi_f)}{N_t}\right)}{a_1t+a_1+L_0} \tag{19}$$

Now, solving eq. (19) yields

$$\begin{aligned}R(t)=&\frac{(a_1+L_0)R(0)+2m\pi_r t}{a_1t+L_0}\\&+2mn(\pi_f+1)\frac{\psi(N_0+t)-\psi(N_0)}{a_1t+L_0}\end{aligned} \tag{20}$$

where $\psi(\cdot)$ represents the digamma function. Note that when there is no random approach for mechanism M3 ($n = 0$), the dynamics of reciprocity do not depend on the number of nodes of the initial network.

To characterize the evolution of clustering, we first need to describe total number of paths of length two at time t. Equation (14) describes the number of new paths at time t (when a new node attaches to the network). The expected number of paths of length two that are formed when a node attaches to the network is captured by eq. (14). Let $a_2 = a_1^2 + m(1 + \pi_f)(n + m\pi_r)$. The total number of paths at time t is P_0+a_2t where P_0 denote the initial number of paths of length two across the entire network. Moreover, eq. (12)

characterizes the expected number of triads that are formed when a new node attaches to the network. The total number of triads at time $t+1$ as a function of $C(t)$ is

$$C(t)(P_0 + a_2 t) + \left(m\pi_f(1+\pi_r) + a_0 \frac{a_1}{N_t} \right) \tag{21}$$

where $a_0 = m(m-1)(2(1+\pi_f)+\pi_r)+n(2(n-1)+m(1+\pi_f+\pi_r))$. The first term in eq. (21) represents the number of triads at time t, and the second term denotes the expected number of new triads at time $t+1$ (according to eq. (12)). The global clustering coefficient at time $t+1$ is

$$C(t+1) = \frac{(P_0 + a_2 t)C(t) + m\pi_f(1+\pi_r) + a_0 a_1 / N_t}{a_2 t + a_2 + P_0} \tag{22}$$

Solving eq. (22) yields

$$\begin{aligned} C(t) = & \frac{(a_2 + P_0)C(0) + m\pi_f(1+\pi_r)t}{a_2 t + P_0} \\ & + a_0 a_1 \frac{\psi(N_0 + t) - \psi(N_0)}{a_2 t + P_0} \end{aligned} \tag{23}$$

Note that if there is no random approach for mechanism M3 and every new node attaches to only one node ($n = 0$ and $m = 1$), then $a_0 = 0$ and the dynamics of clustering do not depend on the number of nodes of the initial network. Moreover, if $L_0, P_0 \ll a_1 < a_2$, then according to eqs. (20) and (23), the response of reciprocity over time is much faster than that of clustering.

Finally, note that eqs. (19) and (22) represent time-varying linear systems of the form

$$y(t+1) = f(t, y_t) \tag{24}$$

where $f(t, y_t) = A(t)y(t) + B(t)$. The following section characterizes the stability properties of both global coefficients based on this closed-loop expression.

VI. Stability of the Global Coefficients

The following theorems describe sufficient conditions for asymptotic stability of systems of the form represented by eq. (24).

Theorem 4 (Theorem 4.2 in [9]): Let the sequence $\{y_t\}$ satisfy a non-autonomous difference equation of the form of eq. (24) with the function $f(t, \cdot)$ satisfying a uniform Lipschitz condition with respect to its second argument for each time t, (with constant $\Gamma_t \leq M < 1$). Then every solution to eq. (24) is asymptotically stable. If the Lipschitz constants satisfy $\Gamma_t \leq 1$ then the conclusion is that every solution is bounded and stable. Further, all solutions are bounded or all are unbounded.

Theorem 5 (Theorem 4.4 in [9]): Consider eq. (24) with solution map operator $y_n = \Phi(n, y_0)$. Assume that, for each t, $\Phi(t, y_0)$ satisfies a uniform Lipschitz condition (with Lipschitz constant Γ_t) with respect to its second argument and the values $\Gamma_t \leq M < \infty$. Then every solution of eq. (24) is stable (but need not be bounded). If, furthermore, $\Gamma_t \to 0$ as $t \to \infty$ then there exists a unique equilibrium solution to eq. (24) and it is asymptotically stable. If, additionally, for every $t \in \mathbb{N}$, $\Gamma_t < t$ for some $|\zeta| < 1$ then the unique equilibrium solution is exponentially stable.

First, to show the stability of the global reciprocity coefficient, we need to show that eq. (19) satisfies the uniform Lipschitz condition. Let

$$f_R(t, R(t)) = \frac{R(t)(L_0 + a_1 t) + b_1(t)}{L_0 + a_1(t+1)}$$

where $b_1(t) = 2m\left(\pi_r + \frac{n(1+\pi_f)}{N_t}\right)$. To show that $f_R(t, R(t))$ satisfies this condition with respect to its second argument, we need to verify that

$$\|f_R(t,x) - f_R(t,y)\| \leq \Gamma_t \|x - y\|$$

Using eq. (19) we get

$$\begin{aligned} & \|f_R(t,x) - f_R(t,y)\| \\ & = \left\| \frac{x(L_0 + a_1 t) + b_1(t)}{L_0 + a_1(t+1)} - \frac{y(L_0 + a_1 t) + b_1(t)}{L_0 + a_1(t+1)} \right\| \\ & = \|x - y\| \left\| \frac{(L_0 + a_1 t)}{L_0 + a_1(t+1)} \right\| \end{aligned}$$

Note that because $\left\| \frac{L_0 + a_1 t}{L_0 + a_1(t+1)} \right\| \leq 1$ for all t we know that $\|f_R(t,x) - f_R(t,y)\| \leq \|x - y\|$, which implies that the Lipschitz condition is satisfied with $\Gamma_t \leq 1$. Applying Theorem 4, we can conclude that every solution to $f_R(t, R(t))$ is bounded and stable.

Now, since eq. (20) is a solution to eq. (19), let

$$\begin{aligned} \Phi_R(t, R(t)) = & \frac{(a_1 + L_0)R(0) + 2m\pi_r t}{a_1 t + L_0} + \\ & 2mn(\pi_f + 1)\frac{\psi(N_0 + t) - \psi(N_0)}{a_1 t + L_0} \end{aligned}$$

In particular, since $\Phi_R(t, R(t))$ does not depend explicitly on $R(t)$, we can assure that it satisfies the Lipschitz condition with respect to the second argument for any fixed t. Moreover, note that as t goes to infinity

$$\lim_{t \to \infty} \frac{\psi(N_0 + t)}{a_1 t + L_0} = 0$$

and

$$\lim_{t \to \infty} \Phi_R(t, R(t)) = \frac{2m\pi_r}{m(1 + \pi_f + \pi_r) + n}$$

which implies that as the network grows in size, the solution $\Phi_R(t, R(t))$ does not depend on $R(t)$ or t. So, the Lipschitz constant $\Gamma_t \to 0$ as $t \to \infty$, and the solution $\Phi_R(t, R(t))$ is asymptotically stable according to Theorem 5. In particular, note that the dynamics of the evolution of the global reciprocity, eq. (19), lead to the theoretical value estimated in eq. (10).

Similarly, to characterize the stability of the stationary clustering coefficient of eq. (22), we will first show that it satisfies the uniform Lipschitz condition. Let

$$f_C(t, C(t)) = \frac{C(t)(P_0 + a_2 t) + b_2(t)}{P_0 + a_2(t+1)}$$

where $b_2(t) = m\pi_f(1+\pi_r) + \frac{a_0 a_1}{N_t}$. To show that $f_C(t, C(t))$

satisfies this condition with respect to its second argument, we need to verify again that $\|f_C(t,x) - f_C(t,y)\| \leq \Gamma_t \|x - y\|$. Using eq. (22) we know that

$$\begin{aligned}
&\|f_C(t,x) - f_C(t,y)\| \\
&= \left\| \frac{x(P_0 + a_2 t) + b_2(t)}{P_0 + a_2(t+1)} - \frac{y(P_0 + a_2 t) + b_2(t)}{P_0 + a_2(t+1)} \right\| \\
&= \|x - y\| \left\| \frac{P_0 + a_2 t}{P_0 + a_2(t+1)} \right\|
\end{aligned}$$

In particular, because $\left\| \frac{P_0 + a_2 t}{P_0 + a_2(t+1)} \right\| \leq 1$ for all t, we know that $\|f_C(t,x) - f_C(t,y)\| \leq \|x - y\|$, which implies that the Lipschitz constant satisfies $\Gamma_t \leq 1$. Applying Theorem 4 we know that $f_C(t, C(t))$ is bounded and stable.

Now, since eq. (23) is a solution to eq. (22), let

$$\begin{aligned}
\Phi_C(t, C(t)) = &\frac{(a_2 + P_0)C(0) + m\pi_f(1 + \pi_r)t}{a_2 t + P_0} \\
&+ a_0 a_1 \frac{\psi(N_0 + t) - \psi(N_0)}{a_2 t + P_0}
\end{aligned}$$

Since the solution Φ_C does not depend on $C(t)$, we know that it satisfies the Lipschitz condition with respect to its second argument for any time t. Moreover, note that as t tends to infinity

$$\lim_{t \to \infty} \frac{\psi(N_0 + t)}{a_2 t + P_0} = 0$$

and

$$\lim_{t \to \infty} \Phi_C(t, C(t)) = \frac{m\pi_f(1 + \pi_r)}{a_1^2 + m(1 + \pi_f)(n + m\pi_r)}$$

which implies that the Lipschitz constant $\Gamma_t \to 0$ as $t \to \infty$. In other words, as the network grows in size, the solution $\Phi_C(t, C(t))$ does not depend on $C(t)$ or t. As a consequence, based on Theorem 5, the solution $\Phi_C(t, C(t))$ is asymptotically stable. In particular, note that the dynamics of the global clustering, eq. (22), lead to the theoretical value estimated in eq. (17).

The above results imply that over time the combination of mechanisms M1-M3 results in network structures with a stationary pair of global reciprocity and clustering coefficients, starting from any initial network.

VII. Simulations

Next, we illustrate the evolution of reciprocity from various initial networks. The solid lines in Figure 1 represent the theoretical value based on eq. (19). The dots indicate the average of 50 simulation runs, starting from an initial network with $N_0 = 20$ (left plot) and $N_0 = 50$ (right plot). The dashed line represents the global reciprocity coefficient, based on eq. (10). In particular, note that for networks with varying initial global reciprocity coefficients, the combination of all three mechanisms leads to the same stationary level of reciprocity. Moreover, note that when the number of initial nodes increases, the evolution of the reciprocity is monotonic.

Finally, Figure 2 illustrates the dynamics of the global clustering coefficient. The solid lines represent the theoretical value based on eq. (22), and the dots indicate the average of 50 simulation runs. The dashed line represents the expected global clustering based on eq. (17). Higher values of the global clustering coefficient are achieved for low values of m, with no random response ($n = 0$).

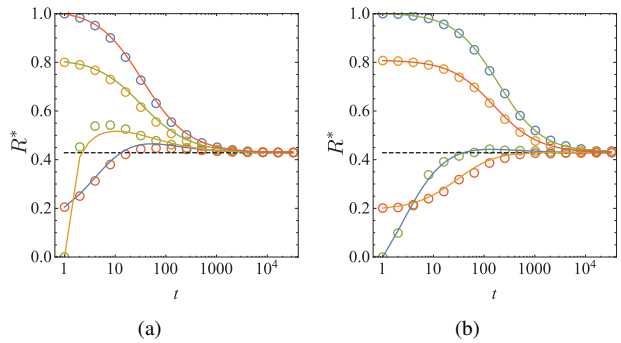

Fig. 1. Evolution of the network reciprocity based on eq. (19); (a) $N_0 = 20$; (b) $N_0 = 50$.

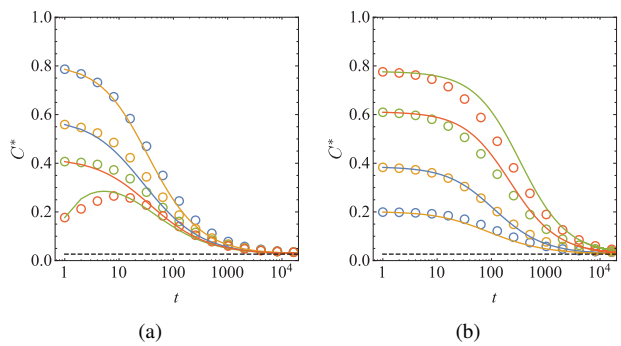

Fig. 2. Evolution of the global clustering based on eq. (22); (a) $N_0 = 20$; (b) $N_0 = 50$.

References

[1] K. Zhao, X. Wang, M. Yu, and B. Gao, "User recommendations in reciprocal and bipartite social networks–an online dating case study," *IEEE Intelligent Systems*, vol. 29, no. 2, pp. 27–35, Mar 2014.

[2] S. L. Johnson, S. Faraj, and S. Kudaravalli, "Emergence of power laws in online communities: The role of social mechanisms and preferential attachment." *MIS Quarterly*, vol. 38, no. 3, pp. 795–808, 2014.

[3] K. Guerrero and J. Finke, "On the formation of community structures from homophilic relationships," in *Proceedings of the American Control Conference*, Montreal, QC, June 2012, pp. 5318–5323.

[4] N. Sastry, D. Manjunath, K. Sollins, and J. Crowcroft, "Data delivery properties of human contact networks," *IEEE Transactions on Mobile Computing*, vol. 10, no. 6, pp. 868–880, 2011.

[5] V. Zlatić and H. Štefančić, "Influence of reciprocal edges on degree distribution and degree correlations," *Physical Review E*, vol. 80, no. 1, p. 016117, 2009.

[6] ——, "Model of Wikipedia growth based on information exchange via reciprocal arcs," *EPL (Europhysics Letters)*, vol. 93, no. 5, p. 58005, 2011.

[7] S. Segarra and A. Ribeiro, "A stable betweenness centrality measure in networks," in *Acoustics, Speech and Signal Processing (ICASSP), 2014 IEEE International Conference on*, May 2014, pp. 3859–3863.

[8] I. Fernandez and J. Finke, "Transitivity of reciprocal networks," in *Proceedings of the IEEE Conference on Decision and Control*, Osaka, Japan, Dec. 2015, pp. 1625–1630.

[9] J. T. Edwards and N. J. Ford, "Boundedness and stability of solutions to difference equations," *Journal of Computational and Applied Mathematics*, vol. 140, no. 1-2, pp. 275 – 289, 2002.