Stability of the Jackson-Rogers model

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***Abstract*— Network formation models describe the dynamics of the structure of connections using mechanisms that operate under different principles. The Jackson-Rogers model is a generic framework that captures the effects of triadic closure in growing networks. Previous results on the model focused on deriving the asymptotic behavior of the node degree distribution based on continuous-time approximations. Here, we use a discrete-time approach to characterize the resulting distribution as an invariant set, and show that this set is asymptotically stable. Furthermore, we show that the expected value of the average degree of the network is also asymptotically stable.**

I. Introduction

Power laws describe the degree distributions of networks found across social sciences, biology, and engineering [1], [2]. To bring insight into the dynamics of the structure of connections that underlie these type of distributions, a common approach has been to model mechanisms that operate under different principles. The principle of preferential attachment refers to a class of mechanisms, in which the newly added node connects to other nodes with a probability that depends on their degree [3]. Based on preferential attachment, the well-known Albert-Barabasi model generates power law degree distributions with scaling exponent $\beta = 3$ (as shown in [4]). Other preferential attachment mechanisms have been proposed, including local and non-linear variations of the original model [5]–[7]. Yet, none of the preferential attachment models is able to recreate the clustering properties often found in empirical networks.

The Jackson-Rogers model generates highly clustered networks with power law degree distributions based on the principle of triadic closure [8]. Three mechanisms underlie the Jackson-Rogers model. First, a newly added node selects m_r nodes and connects to each of them with probability p_r. Second, of the union of the neighbors of these nodes, the new node establishes a link to m_n nodes with probability p_n. Third, $k_0 \geq 0$ nodes establish links to the new node. Using a continuos-time approximation, the authors in [8] show that for a large in-degree, the resulting in-degree distribution follows a power law with scaling exponent $\beta = 2 + \frac{p_r m_r}{p_n m_n}$. Furthermore, as $\frac{p_r m_r}{p_n m_n} \to 0$, the Jackson-Rogers model can be viewed as an indirect preferential attachment mechanism, in which nodes with a high in-degree have a higher probability of establishing links with the new node.

There are a number of variations to the Jackson-Rogers model [9], [10]. The work in [10] analyzes the effects of reciprocity by adding a mechanism that establishes reciprocal links when new nodes attach to the network. The authors show that the global reciprocity coefficient and the global clustering coefficient are both asymptotically stable. However, none of the results on the Jackson-Rogers model focus on the stability properties of the node degree distribution.

The contributions of this paper are the following. First, using a discrete-time approach, we characterize the probability that a random selected node has a particular in-degree. Second, we show that the dynamics of the degree distribution converge. This result holds for any network model where the number of edges grows linearly over time. We derive an expression for the complementary cumulative in-degree distribution. Unlike [11], we consider a positive in-degree for new nodes that attaches to the network (i.e., we allow $k_0 \neq 0$ for the third mechanism). Third, we characterize the complementary cumulative degree distribution as an invariant set and show that this set is asymptotically stable. Finally, we show that the expected average in-degree is also asymptotically stable.

The remainder of this paper is organized as follows. Section II shows the existence of the limit and characterizes the complementary cumulative in-degree distribution. Sections III and IV present the stability properties of the model. Section V presents simulation results. Section VI draws some conclusions and future work.

II. Asymptotic behavior of the in-degree distribution

Consider a sequence $\mathcal{G} = \{\mathcal{G}_0, \mathcal{G}_1, \ldots\}$ where each element $\mathcal{G}_t = (V_t, E_t)$ represents a directed graph at time $t \in \mathbb{N}_0$. V_t represents the set of nodes and $E_t \subseteq V_t \times V_t$ represents the set of edges. A pair $(u, v) \in E_t$ indicates the existence of an edge from node u to node v at time t.

Let K_t denote a random variable that characterizes the in-degree of a node, selected uniformly at random at time t, and let $P_t(k) = P(K_t = k)$ denote the probability that K_t equals k (i.e., the probability that the node has in-degree k at time t). The cumulative distribution function (cdf) of the in-degree of the nodes of $\mathcal{G}_t$ is denoted by

$$F_t(k) = P(K_t < k) = \sum_{x<k} P_t(x)$$

Moreover, $F_t^{c}(k) = P(K_t \geq k) = 1 - F_t(k)$ denotes the complementary cumulative distribution function at time t.

The evolution of the Jackson-Rogers (JR) model is driven by three mechanisms [8].

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M1 *Random attachment:* A newly added node chooses m_r nodes, selected uniformly at random from the set of nodes in V_{t-1}, and connects with each of them with probability p_r.

M2 *Triadic closure:* Of the union of the outgoing neighbors of the m_r nodes, the new node chooses m_n nodes, selected uniformly at random, and connects with each of them with probability p_n.

M3 *Network response:* A total of $k_0 \geq 0$ nodes in V_{t-1} connect to the new node.

Mechanism M1 indicates that the new node tries to connect to m_r nodes. Note that the neighbors of these nodes are the targets of mechanism M2. Note also that mechanism M3 guarantees that the in-degree of new nodes is equal to k_0.

Next, let M_t represent a random variable that characterizes the number of edges established by a new node at time t. Because mechanisms M1 and M2 follow Bernoulli processes, M_t follows a binomial distribution and thus the expected value $\mathrm{E}[M_t]$ exists. We know that $\mathrm{E}[M_t] = p_r m_r + p_n m_n$. Let $m = \mathrm{E}[M_t]$ and $n_t = |V_t|$. Furthermore, let $\pi_t(k)$ denote the probability that a new node connects to node i of in-degree k. We know that

$$\pi_t(k) = p_r \frac{m_r}{n_{t-1}} + p_n \frac{\sum_{i=1}^{k} \binom{k}{i} \binom{n_{t-1}-k}{m_r-i}}{\binom{n_{t-1}}{m_r}} \frac{m_n}{m_r(m+k_0)} \quad (1)$$

Note that $\pi_t(k)$ depends only on mechanisms M1 and M2. The first term on the right-hand side of eq. (1) represents the probability that the new node connects to a node of in-degree k through mechanism M1. The second term represents the probability that a new node connects to a node of in-degree k through mechanism M2. Note that the expression

$$\frac{\sum_{i=1}^{k} \binom{k}{i} \binom{n_{t-1}-k}{m_r-i}}{\binom{n_{t-1}}{m_r}}$$

corresponds to the probability of choosing a parent of a node of in-degree k, and

$$\frac{m_n}{m_r(m+k_0)}$$

is the probability of choosing such a node of in-degree k to form a triad [12]. If $n_{t-1} \gg m_r k$, an approximation for eq. (1) is given by [8]

$$\pi_t(k) = p_r \frac{m_r}{n_{t-1}} + p_n \frac{k m_n}{n_{t-1}(m+k_0)} \quad (2)$$

To specify the cdf of the in-degree, we need to guarantee the existence of $\lim_{t\to\infty} P_t(k)$ for all $k \geq k_0$, which can be shown by establishing an asymptotic equivalence between two sequences[1]. Consider the following lemma.

Lemma 1: Let $\{s_t\}, \{u_t\}, \{v_t\}$ and $\{w_t\}$ be sequences of non-negative real numbers such that $s_t \sim u_t$ and $v_t \sim w_t$.

1) If $\lim_{t\to\infty} u_t = L < \infty$, then $\lim_{t\to\infty} s_t = L$.
2) $s_t + v_t \sim u_t + w_t$.

Proof:

1) If $s_t \sim u_t$, then we know that $\lim_{t\to\infty} s_t/u_t = 1$. That is, for all $\varepsilon > 0$ there exists $T_1 \in \mathbb{N}$ such that for all $t > T_1$, $\left|\frac{s_t}{u_t} - 1\right| < \varepsilon$. Now, if $\lim_{t\to\infty} u_t = L$, then for all $\varepsilon > 0$ there exists $T_2 \in \mathbb{N}$ such for all $t > T_2$, $|u_t - L| < \varepsilon$. Furthermore, note that

$$\begin{aligned} |s_t - L| &\leq |s_t - u_t| + |u_t - L| \\ &< |s_t - u_t| + \varepsilon \\ &= |u_t| \left|\frac{s_t}{u_t} - 1\right| + \varepsilon \\ &< (L+\varepsilon)\varepsilon + \varepsilon = \varepsilon' \end{aligned}$$

Therefore, for all $\varepsilon' > 0$, there exists $T = \max\{T_1, T_2\}$ such that for all $t > T$, $|s_t - L| < \varepsilon'$.

2) Because $\{s_t\}, \{u_t\}, \{v_t\}$ and $\{w_t\}$ are sequences of non-negative real numbers, note that if $\frac{s_t}{u_t} < \frac{v_t}{w_t}$, then

$$\frac{s_t}{u_t} < \frac{s_t + v_t}{u_t + w_t} < \frac{v_t}{w_t}$$

Because $s_t \sim u_t$ and $v_t \sim w_t$, applying the Squeeze Theorem, we have that

$$\lim_{t\to\infty} \frac{s_t + v_t}{u_t + w_t} = 1$$

That is, $s_t + v_t \sim u_t + w_t$. The same analysis applies for $\frac{v_t}{w_t} \leq \frac{s_t}{u_t}$.

■

We use Lemma 1 to characterize the asymptotic behavior of the in-degree distribution of the network, under the following assumption on the initial network.

A1 The initial number of nodes satisfies $n_0 > m_r + m_n$.

Assumption A1 guarantees that the first new node can connects to up to $m_r + m_n$ nodes in $\mathcal{G}_0$.

Theorem 1: Under assumption A1, the limit of $P_t(k)$ as t tends to infinity exists for all $k \geq k_0$.

Proof: First, we determine a recursive expression for $P_t(k)$ for all $k \geq k_0$. Note that only mechanisms M1 and M2 can alter the probability that a randomly selected node has a particular in-degree k at time t. According to eq. (2), we know that the expected number of nodes of in-degree k to which a new node connects at time t is

$$\pi_t(k) n_{t-1} P_{t-1}(k) = \left(p_r m_r + \frac{k p_n m_n}{m+k_0}\right) P_{t-1}(k) \quad (3)$$

Using eq. (3), we have that the expected number of nodes of in-degree $k > k_0$ at time t is

$$\begin{aligned} n_t P_t(k) &= n_{t-1} P_{t-1}(k) - \pi_t(k) n_{t-1} P_{t-1}(k) \\ &\quad + \pi_t(k-1) n_{t-1} P_{t-1}(k-1) \\ &= n_{t-1} P_{t-1}(k) - \left(p_r m_r + \frac{k p_n m_n}{m+k_0}\right) P_{t-1}(k) \\ &\quad + \left(p_r m_r + \frac{(k-1) p_n m_n}{m+k_0}\right) P_{t-1}(k-1) \quad (4) \end{aligned}$$

In other words, the expected number of nodes of in-degree $k > k_0$ at time t is equal to the expected number of nodes of in-degree k at time $t-1$, minus the expected number of nodes of in-degree k at time $t-1$ selected by either M1 or

[1]Two sequences $\{s_t\}$ and $\{u_t\}$ are equivalent, denoted by $s_t \sim u_t$, if $\lim_{t\to\infty} s_t/u_t = 1$.

M2, plus the expected number of nodes of in-degree $k-1$ that connects the new node.

Now, because k_0 nodes establish an edge to the new node, the expected number of nodes of in-degree $k=k_0$ at time t is

$$\begin{aligned} n_t P_t(k_0) &= n_{t-1}P_{t-1}(k_0) - \pi_t(k_0) n_{t-1} P_{t-1}(k_0) + 1 \\ &= n_{t-1}P_{t-1}(k_0) - \left(p_r m_r + \frac{k_0 p_n m_n}{m+k_0}\right) P_{t-1}(k_0) \\ &\quad + 1 \end{aligned} \tag{5}$$

The first term represents the expected number of nodes of in-degree k_0 at time $t-1$. The second term corresponds to the expected number of nodes of in-degree k_0 that connect with the new node at time t. The number 1 represents the new node that attaches to the network with in-degree k_0.

Second, using eqs. (4) and (5), we proceed by induction over k to guarantee the existence of $\lim_{t\to\infty} P_t(k)$. Consider the two base cases when $k = k_0$ and $k = k_0 + 1$.[2] Using eq. (5), note that $P_t(k_0)$ can be expressed using the recurrence

$$P_t(k_0) = \frac{1}{n_t}\left(\left(n_{t-1} - p_r m_r - \frac{k_0 p_n m_n}{m+k_0}\right) P_{t-1}(k_0) + 1\right)$$

with initial condition $P_0(k_0) = q$, for any $q \in [0,1]$. Letting $b = -1 - p_r m_r - \frac{k_0 p_n m_n}{m+k_0}$, and using mathematical induction, it can be shown that for all $t \geq 0$

$$P_t(k_0) = \frac{1}{b}\left(-1 + \frac{(1+bq)\Gamma(n_0+1)\Gamma(n_0+1+b+t)}{\Gamma(n_0+1+b)\Gamma(n_0+1+t)}\right) \tag{6}$$

According to assumption A1, note that eq. (6) is well-defined ($n_0 + 1 + b + t > 0$ for all $t \in \mathbb{N}_0$). Because $b < 0$, note also that

$$\Gamma(n_0 + 1 + t + \lfloor b \rfloor) \leq \Gamma(n_0 + 1 + t + b) \leq \Gamma(n_0 + t)$$

where $\lfloor \cdot \rfloor$ denotes the floor function. Applying the Squeeze Theorem, we have

$$\lim_{t\to\infty} \frac{\Gamma(n_0+1+b+t)}{\Gamma(n_0+1+t)} = 0$$

and thus

$$\lim_{t\to\infty} P_t(k_0) = \frac{m+k_0}{(m+k_0)(1+p_r m_r) + k_0 p_n m_n}$$

Next, consider $k = k_0 + 1$. Using eq. (4) we have that $n_t P_t(k_0+1)$ equals to

$$\begin{aligned} &\left(n_{t-1} - p_r m_r - \frac{(k_0+1)p_n m_n}{m+k_0}\right) P_{t-1}(k_0+1) \\ &+ \left(p_r m_r + \frac{k_0 p_n m_n}{m+k_0}\right) P_{t-1}(k_0) \end{aligned} \tag{7}$$

Because the number of outgoing edges of new nodes is less than n_t, note that for a large enough t, $P_{t-1}(k_0) \sim P_t(k_0)$ and $P_{t-1}(k_0+1) \sim P_t(k_0+1)$. Moreover, using Lemma 1, we know from eq. (7) that for a large enough t

$$\begin{aligned} &\left(1 + p_r m_r + \frac{(k_0+1)p_n m_n}{m+k_0}\right) P_t(k_0+1) \\ &\sim \left(p_r m_r + \frac{k_0 p_n m_n}{m+k_0}\right) P_t(k_0) \end{aligned}$$

which guarantees that $\lim_{t\to\infty} P_t(k_0+1)$ equals to

$$\frac{(m+k_0)p_r m_r}{(m+k_0)(1+p_r m_r) + p_n m_n} \lim_{t\to\infty} P_t(k_0)$$

Now, assume that $\lim_{t\to\infty} P_t(\ell)$ exists. Using eq. (4) with $k = \ell + 1$, we have that $n_t P_t(\ell+1)$ satisfies

$$\begin{aligned} &\left(n_{t-1} - p_r m_r - \frac{(\ell+1)p_n m_n}{m+k_0}\right) P_{t-1}(\ell+1) \\ &+ \left(p_r m_r + \frac{\ell p_n m_n}{m+k_0}\right) P_{t-1}(\ell) \end{aligned} \tag{8}$$

Because for a large enough t, $P_{t-1}(\ell) \sim P_t(\ell)$ and $P_{t-1}(\ell+1) \sim P_t(\ell+1)$, using Lemma 1 and eq. (8), we have

$$\begin{aligned} &\left(1 + p_r m_r + \frac{(\ell+1)p_n m_n}{m+k_0}\right) P_t(\ell+1) \\ &\sim \left(p_r m_r + \frac{\ell p_n m_n}{m+k_0}\right) P_t(\ell) \end{aligned}$$

Moreover, based on the inductive hypothesis, we know that $\lim_{t\to\infty} P_t(\ell)$ exists, so $\lim_{t\to\infty} P_t(\ell+1)$ also exists and is equal to

$$\frac{(m+k_0)p_r m_r + \ell p_n m_n}{(m+k_0)(1+p_r m_r) + (\ell+1)p_n m_n} \lim_{t\to\infty} P_t(\ell)$$

Therefore, $\lim_{t\to\infty} P_t(k)$ exists for all $k \geq k_0$. ■

As a consequence of Theorem 1, the next corollary characterizes the in-degree distribution of the nodes of the network.

Corollary 1: Suppose assumption A1 holds. For $k \geq k_0$, the asymptotic behavior of the expected complementary cumulative in-degree distribution $F^c_\infty(k)$ is

$$\frac{\Gamma(k + a(k_0+m))\Gamma(k_0 + (a + (p_n m_n)^{-1})(k_0+m))}{\Gamma(k_0 + a(k_0+m))\Gamma(k + (a + (p_n m_n)^{-1})(k_0+m))}$$

for $p_n m_n \neq 0$ and $a = \frac{p_r m_r}{p_n m_n}$, and

$$1 - \left(\frac{p_r m_r}{1 + p_r m_r}\right)^{k-k_0}$$

for $p_n m_n = 0$.

Proof: Let $P_\infty(k)$ denote $\lim_{t\to\infty} P_t(k)$ for all $k \geq k_0$. Using Theorem 1, we know that

$$P_\infty(k) = \frac{(m+k_0)p_r m_r + (k-1)p_n m_n}{(m+k_0)(1+p_r m_r) + k p_n m_n} P_\infty(k-1)$$

for all $k > k_0$, and

$$P_\infty(k) = \frac{m+k_0}{(m+k_0)(1+p_r m_r) + k_0 p_n m_n}$$

for $k = k_0$. Note that for $k > k_0$, $P_\infty(k)$ is defined as a recurrence function with initial condition $P_\infty(k_0)$. Following

[2] Because k_0 is the lowest in-degree, note that $k = k_0 + n$ for all $n \in \mathbb{N}_0$. This implies that we do an inductive analysis over all $\mathbb{N}_0$.

an inductive process, it can be shown that

$$P_\infty(k) = \tfrac{k_0+m}{(k_0+m)p_r m_r + k p_n m_n} \prod_{\ell=k_0}^{k} \tfrac{(k_0+m)p_r m_r + \ell p_n m_n}{(k_0+m)(1+p_r m_r)+\ell p_n m_n}$$

For $p_n m_n \neq 0$ and letting $a = \frac{p_r m_r}{p_n m_n}$, using the gamma function, we know that

$$P_\infty(k) = \tfrac{(k_0+m)\Gamma(k+a(k_0+m))\Gamma(k_0+(k_0+m)(a+(p_n m_n)^{-1}))}{p_n m_n \Gamma(k_0+a(k_0+m))\Gamma(k+1+(k_0+m)(a+(p_n m_n)^{-1}))} \tag{9}$$

For $p_n m_n = 0$, we have

$$P_\infty(k) = \frac{1}{1+p_r m_r}\left(\frac{p_r m_r}{1+p_r m_r}\right)^{k-k_0} \tag{10}$$

Because $F^c_\infty(k) = P[K_\infty \geq k] = 1 - \sum_{\ell=k_0}^{k-1} P_\infty(\ell)$, using eqs. (9) and (10) we obtain the desired results. ■

Next, we characterize the expected average in-degree of the model. Let d_0 represent the total in-degree of the initial network, i.e., the sum of the in-degree of each node of $\mathcal{G}_0$. Let A_t denote a random variable that represents the average in-degree of the nodes in $\mathcal{G}_t$. Note that

$$A_t = \frac{d_0 + tk_0 + \sum_{i=1}^t M_i}{n_0 + t}$$

Because $\mathrm{E}[M_t] = m$ for all t, if $X_t = \frac{1}{t}\sum_{i=1}^t M_i$, using the Law of Large Numbers, we know that $\lim_{t\to\infty} X_t = m$. So we have

$$\begin{aligned}\lim_{t\to\infty} A_t &= \lim_{t\to\infty} \frac{d_0 + tk_0 + \sum_{i=1}^t M_i}{n_0 + t}\\ &= \lim_{t\to\infty} \frac{tk_0 + \sum_{i=1}^t M_i}{t}\\ &= k_0 + m\end{aligned}$$

Based on the linearity properties of the expected value, note that

$$\begin{aligned}\mathrm{E}[A_t] &= \mathrm{E}\left[\frac{d_0 + tk_0 + \sum_{i=1}^t M_i}{n_0 + t}\right]\\ &= \frac{d_0 + t(k_0+m)}{n_0+t}\end{aligned} \tag{11}$$

which implies that $\lim_{t\to\infty}\mathrm{E}[A_t] = k_0 + m$. That is, $A_t \sim \mathrm{E}[A_t]$.

Remark 1: Note that the monotonicity of the expected average in-degree depends on the total in-degree of the initial network $\mathcal{G}_0$. If $d_0 < n_0(k_0+m)$, then $\mathrm{E}[A_t]$ is strictly increasing; if $d_0 > n_0(k_0+m)$, then $\mathrm{E}[A_t]$ is strictly decreasing; and if $d_0 = n_0(k_0+m)$, then $\mathrm{E}[A_t]$ is equal to k_0+m for all $t \geq 0$.

In the next section, we use the expected average in-degree of the network to characterize the stability properties of the complementary cumulative in-degree distribution.

III. Stability of the Complementary Cumulative In-Degree Distribution

Define the state of the network at time t as an infinite dimensional vector $x_t = (x_t(0), x_t(1), \ldots)$, where $x_t(k)$ represents the probability that a node in $\mathcal{G}_t$ has an expected in-degree greater than or equal to k, that is, for all $k \in \mathbb{N}_0$, $x_t(k) = P[\mathrm{E}[K_t] \geq k]$. Let $\mathcal{R} = [0,1]^\infty$. The state space $\mathcal{X}$ is defined as

$$\mathcal{X} = \left\{ x \in \mathcal{R} : \sum_{k=0}^\infty x(k) = m + k_0 + 1 \Rightarrow x(k) = F^c_\infty(k)\right\}$$

Let $x^e \in \mathcal{X}$ denote the equilibrium distribution of the model, defined as

$$x^e = (x^e(0), x^e(1), \ldots)$$

where $x^e(k) = F^c_\infty(k)$. According to Corollary 1, we know that $x^e(k) > 0$ for all $k \geq 0$. Moreover, note that

$$\begin{aligned}\sum_{k=0}^\infty x_t(k) &= \sum_{k=0}^\infty (k+1)P(\mathrm{E}[K_t]=k)\\ &= \sum_{k=0}^\infty kP(\mathrm{E}[K_t]=k) + \sum_{k=0}^\infty P(\mathrm{E}[K_t]=k)\\ &= \mathrm{E}[A_t]+1\end{aligned}$$

Using eq. (11) we have

$$\lim_{t\to\infty}\sum_{k=0}^\infty x_t(k) = \sum_{k=0}^\infty x^e(k) = k_0 + m + 1$$

For all $k \in \mathbb{N}_0$, consider the set

$$\mathcal{X}_C = \left\{ x \in \mathcal{X} : \sum_{k=0}^\infty x(k) = k_0 + m + 1\right\} \tag{12}$$

It is clear that $x^e \in \mathcal{X}_C$. Moreover, for any finite $t \in \mathbb{N}$, there exists a $k \in \mathbb{N}$ such that $x_t(k) = 0$, i.e., $x_t(k) \notin \mathcal{X}_C$. Thus, x^e is the unique state in $\mathcal{X}_C$. Furthermore, because $\lim_{t\to\infty} x_t(k) = F^c_\infty(k)$, the set $\mathcal{X}_C$ corresponds to a positive limit set of the model and, according to Lemma 3.1 in [13], it is an invariant set.

To guarantee that $\mathcal{X}_C$ is asymptotically stable, we introduce the following distance function on $\mathcal{X}$. Let $\rho : \mathcal{X}\times\mathcal{X} \to \mathbb{R}^+_0$ be defined as

$$\rho(x,y) = \left|\sum_{k=1}^\infty (x(k) - y(k))\right|$$

It is easy to show that ρ is a pseudometric on $\mathcal{X}$ and thus the ordered pair $(\mathcal{X}, \rho)$ a pseudometric space[3].

Theorem 2: The invariant set $\mathcal{X}_C$ is asymptotically stable.

Proof: Let $\mathcal{V}(x) = \rho(x, x^e)$ be a Lyapunov candidate function. Because x^e is the only state that satisfies the equation $\sum_{k=0}^\infty x(k) = k_0 + m + 1$, note that for all $x \in \mathcal{X}$ with $x \neq x^e$, $\mathcal{V}(x) > 0$. Furthermore, it is clear that $\mathcal{V}(x^e) = 0$. The following four conditions are sufficient to guarantee the asymptotic stability of $\mathcal{X}_C$ [14].

[3] Unlike a metric space, in a pseudometric space $(\mathcal{X}, \rho)$, for $x, y \in \mathcal{X}$, $\rho(x,y) = 0$ does not necessarily implies that $x = y$.

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Existence of a lower bound: From the definition of $\mathcal{V}$, we have that for all sufficiently small $\varepsilon_1 > 0$, there exists a $\delta_1 = \varepsilon_1$ such that for all $x \in \mathcal{X}$, if $\rho(x, x^e) > \varepsilon_1$, then $\mathcal{V}(x) > \delta_1$.

Existence of an upper bound: Note that for all sufficiently small $\varepsilon_2 > 0$, there exists a $\delta_2 = \varepsilon_2$ such that for all $x \in \mathcal{X}$, if $\rho(x, x^e) < \delta_2$, then $\mathcal{V}(x) \leq \varepsilon_2$.

Non-increasing along all possible motions of the model: Consider the following two cases based on the total degree d_0 of the initial network $\mathcal{G}_0$. According to Remark 1, if $d_0 < n_0(k_0 + m)$, then

$$\begin{aligned}\mathcal{V}(x_t) &= \left|\sum_{k=0}^{\infty} x_t(k) - \sum_{k=0}^{\infty} x^e(k))\right| \\ &= m + k_0 + 1 - \sum_{k=0}^{\infty} x_t(k)\end{aligned}$$

Because $\sum_{k=0}^{\infty} x_t(k)$ converges to $k_0 + m + 1$ from below, we have that

$$\mathcal{V}(x_t) - \mathcal{V}(x_{t+1}) = \sum_{k=0}^{\infty} x_{t+1}(k) - \sum_{k=0}^{\infty} x_t(k) > 0$$

Similarly, if $d_0 < n_0(k_0 + m)$ then

$$\mathcal{V}(x_t) - \mathcal{V}(x_{t+1}) = \sum_{k=0}^{\infty} x_t(k) - \sum_{k=0}^{\infty} x_{t+1}(k) > 0$$

Convergence: Because $x_t(k) = P[\mathrm{E}[K_t] \geq k] = \mathrm{E}[F_t^c(k)]$, then $\lim_{t\to\infty} x_t(k) = \bar{F}_\infty(k)$, which implies that $\mathcal{V}(x_t) \to 0$ as $t \to \infty$. ■

The next section characterizes a family of invariant sets that capture the asymptotic behavior of the expected average in-degree of the network. We show that these sets are asymptotically stable and that the trajectories of the model are uniformly ultimate bounded in the mean with respect to the average degree.

IV. Stability of the Average In-Degree

Denote $y_t = \mathrm{E}[A_t]$ and let $y^e = k_0 + m$. Define the state space as the set of all positive real numbers including zero, that is, $\mathcal{Y} = \mathbb{R}_0^+$. Let $d(\cdot, \cdot)$ denote the Euclidean distance. For $\varepsilon \geq 0$ define the set

$$\mathcal{Y}^\varepsilon = \begin{cases} \{y \in \mathcal{Y} : d(y, y^e) \leq \varepsilon\} & \text{if } \varepsilon > 0 \\ \\ \{y \in \mathcal{Y} : d(y, y^e) = 0\} & \text{if } \varepsilon = 0 \end{cases}$$

The following theorem characterizes the invariance of $\mathcal{Y}^\varepsilon$.

Theorem 3: The set $\mathcal{Y}^\varepsilon$ is invariant for all $\varepsilon \geq 0$.

Proof: First, consider $\varepsilon > 0$. Since $\lim_{t\to\infty} y_t = y^e$, there exists a positive integer $T \geq \frac{|d_0 - n_0 y^e| - \varepsilon n_0}{\varepsilon}$ such that for all $t > T$, $d(y_t, y^e) < \varepsilon$. That is, if a trajectory of the expected average in-degree of the network belongs to $\mathcal{Y}^\varepsilon$ at some time t, then it remains in its interior for all $t' > t$. Now, if $d(y_t, y^e) = \varepsilon$, according to the monotonicity of $\mathrm{E}[A_t]$ (Remark 1), the trajectory of y_t at the time $t + 1$ will enter to the interior of $\mathcal{Y}^\varepsilon$. Therefore, $\mathcal{Y}^\varepsilon$ is an invariant set for all $\varepsilon > 0$.

Second, consider $\varepsilon = 0$. It is clear that $\mathcal{Y}^0 = \{y^e\}$. Suppose that there exists a time t such that $y_t = y^e$, that is, $\mathrm{E}[A_t] = k_0 + m$. Without loss of generality, define $\mathcal{G}'_{t_0} = \mathcal{G}_t$ as an initial network of the model. Using again Remark 1, we know that $\mathrm{E}[A_{t'}]$ is constant for all $t' \geq t_0 = t$ and is equal to $k_0 + m$. That is, $y_{t'} = y^e$ for all $t' > t$, which implies that $\mathcal{Y}^0$ is invariant. ■

Next, we show that the motions of the JR model are uniformly ultimately bounded in the mean with respect to the invariant set $\mathcal{Y}^0$ [15], [16].

Theorem 4: The trajectories of the expected average in-degree of the network are uniformly ultimately bounded in the mean with respect to the invariant set $\mathcal{Y}^0$.

Proof: Let $\varepsilon > 0$ fixed and let $B = \varepsilon$. For $y_0 = \frac{d_0}{n_0}$, suppose that $d(y_0, y^e) < \alpha$ for any $\alpha > 0$. Consider two cases for y_0. First, assume that $y_0 > k_0 + m$. Note that

$$\frac{|d_0 - n_0(k_0 + m)| - Bn_0}{B} = \frac{n_0}{d_0}\lambda_1$$

where $\lambda_1 = \frac{d_0/n_0 - B - k_0 - m}{B/d_0}$. We know that $d(y_t, y^e) < B$ for any $t > \frac{n_0}{d_0}\lambda_1$. Furthermore, because $d(y_0, y^e) < \alpha$, we have that

$$t > \frac{n_0}{d_0}\lambda_1 > \frac{\lambda_1}{\alpha + k_0 + m} = T(\alpha)$$

Second, if $y_0 < k_0 + m$, by following a similar analysis as for $y_0 > k_0 + m$, we know that if $d(y_0, y^e) < \alpha$, then

$$t > \frac{n_0}{d_0}\lambda_2 > \frac{\lambda_2}{\alpha + k_0 + m} = T(\alpha)$$

where $\lambda_2 = \frac{k_0 + m - B - d_0/n_0}{B/d_0}$. In any case, there exists $B > 0$ and corresponding to any $\alpha > 0$, there exists $T(\alpha) > 0$ such that if $d(y_0, y^e) < \alpha$, then $d(y_t, y^e) < B$ for all $t > T(\alpha)$. ■

Next, to characterize the asymptotic stability of the invariant sets $\mathcal{Y}^\varepsilon$, we consider the following function

$$\bar{\rho}(y, \mathcal{Y}^\varepsilon) = \inf\{d(y, u) : u \in \mathcal{Y}^\varepsilon\}$$

Note that for any $\varepsilon \geq 0$ and any $y \in \mathcal{Y}$, $\bar{\rho}(y, \mathcal{Y}^\varepsilon)$ satisfies

1) $\bar{\rho}(y, \mathcal{Y}^\varepsilon) \geq 0$; and
2) $\bar{\rho}(y, \mathcal{Y}^\varepsilon) = 0$ if and only if $y \in \mathcal{Y}^\varepsilon$.

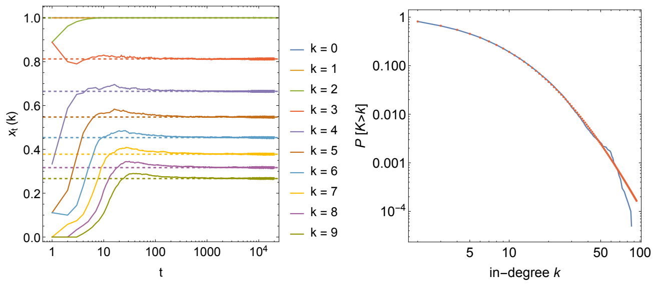
Theorem 5: For a fixed $\varepsilon \geq 0$, $\mathcal{Y}^\varepsilon$ is asymptotically stable.

Proof: Let $\mathcal{V}(y) = \bar{\rho}(y, \mathcal{Y}^\varepsilon)$ be a Lyapunov candidate function. Using Remark 1 and by following a similar analysis as in Theorem 2, we obtain the result. ■

V. Simulations

This section presents some simulations that illustrate our theoretical results. Let $m_r = 5$, $m_n = 3$, $p_r = 0.8$, $p_n = 0.4$ and $k_0 = 2$. Let $\mathcal{G}_0 = (V_0, E_0)$ be an initial network with $|V_0| = 9$ and in-degrees $(3, 3, 3, 3, 1, 3, 6, 4, 4)$. Note that $d_0 = 30$. The initial state is given by

$$x_0 = \{1, 1, 0.88, 0.88, 0.33, 0.11, 0.11, 0, 0, \ldots\}$$



(a) (b)

Fig. 1. (a) Evolution of $x_t(k) = P[\mathrm{E}[K_t] \geq k]$ for $0 \leq k_0 \leq 9$; and (b) Complementary cumulative in-degree distribution. Solid line represents the average of the ccdf of 100 runs of the model and the dashed represents the predictions for $m_r = 5$, $m_n = 3$, $p_r = 0.8$, $p_n = 0.4$, $k_0 = 2$ and $d_0 = 30$.

According to eq. (12), the invariant set $\mathcal{X}_C$ is given by

$$\mathcal{X}_C = \{x \in \mathcal{X} : \textstyle\sum_{k=0}^{\infty} x(k) = 8.2\}$$

Figure 1 illustrates the evolution of the states of the model for the in-degree together with the theoretical values of $x^e(k)$ for $0 \leq k \leq 9$. Note that the simulated distributions approach the theoretical limits (based on Corollary 1). Simulations correspond to an average of 100 runs of the model. Furthermore, Figure 2 shows the evolution of the Lyapunov function $\mathcal{V}(x) = \rho(x, x^e)$. Note that, as shown in Theorem 2, $\mathcal{V}$ is a strictly decreasing function and converges to zero as $t \to \infty$.

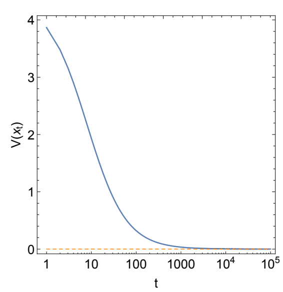


Fig. 2. Behavior of the Lyapunov function $\mathcal{V}(x) = \rho(x, x^e)$.

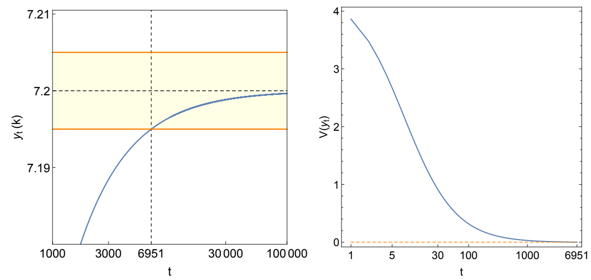
Now, to illustrate the results of Section IV, consider $\varepsilon =$ 0.005. Define the invariant set

$$\mathcal{Y}^\varepsilon = \{y \in \mathbb{R}_0^+ : d(y, y^e) \leq 0.005\}$$

According to Theorem 3, a trajectory of the evolution of the average in-degree enters to $\mathcal{Y}^\varepsilon$ at $T = \left\lceil \frac{|d_0 - n_0 y^e| - \varepsilon n_0}{\varepsilon} \right\rceil =$ 6951 and remains in its interior for $t > T$ (see Figure 3(a)). Figure 3(b) illustrates the behavior of the Lyapunov function $\mathcal{V}(y) = \bar{\rho}(y, \mathcal{Y}^\varepsilon)$ that guarantees the asymptotic stability of $\mathcal{Y}^\varepsilon$.

VI. Conclusions

Based on the discrete version of the Jackson-Rogers model, our work uses a discrete-time approach to characterize the evolution of the probability function that a node has particular in-degree k at any time t. Moreover,



(a) (b)

Fig. 3. (a) Illustration of the instant at which a trajectory that represents the evolution of the expected average in-degree enters the invariant set $\mathcal{Y}^\varepsilon$; and (b) behavior of the Lyapunov function $\mathcal{V}(y) = \bar{\rho}(y, \mathcal{Y}^\varepsilon)$ that guarantees the stability of $\mathcal{Y}^\varepsilon$.

we describe the asymptotic behavior for the cumulative in-degree distribution, and show that the distribution is indeed asymptotically stable. We also show that the average in-degree is asymptotically stable and that the motions of the expected average in-degree of the model are uniformly ultimately bounded in the mean with respect to the invariant set that represents the asymptotic average. Characterizing the stability properties of other centrality measures, for example, the eigenvector centrality, remains a future research direction.

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