Supplement to "Dynamics of Group Cohesion in Homophilic Networks"

Proof of Lemma 1. Consider an event $e_i \in g_e(x)$ in which agent i disconnects from agent j and connects to agent k. The probability that the event is of type 1 is $P[e_i \in E_1] = \varepsilon$ (and the probability that the event is of type 2 is $P[e_i \in E_2] = 1 - \varepsilon$).

If $e_i \in E_1$, note that x_i must be strictly less than 1. The probability that agent i disconnects from an agent of the same type is

$$P[p_i = p_j | e_i \in E_1] = 0 \tag{13}$$

Moreover, the probability that agent i connects to an agent of the same type is

$$P[p_i = p_k | e_i \in E_1] = \frac{n_\ell - |Q_i'| - 1}{n - q - 1} = \frac{n_\ell - qx_i - 1}{n - q - 1} \tag{14}$$

where $n-q-1$ represents the total number of non-neighboring agents of agent i, and $n_\ell - |Q_i'| - 1$ the number of non-neighboring agents that are of the same type. Equation (14) captures how group size impacts the probability that agent i connects to an agent of the same type. Note that when agent i establishes a link, the probability that the new neighbor is of the same type depends in general on n, n_ℓ, q and x_i.

Next, if $e_i \in E_2$, then x_i is less than or equal to 1. The probability that agent i disconnects from an agent of the same type is

$$P[p_i = p_j | e_i \in E_2] = \frac{|Q_i'|}{q} = x_i \tag{15}$$

Note that (15) only depends on x_i, but not on n or n_ℓ. In other words, group size does not impact the probability that agent i disconnects from an agent of the same type. Moreover, the probability that agent i connects to an agent of the same type is

$$P[p_i = p_k | e_i \in E_2] = 0 \tag{16}$$

According to Assumption 1(b), if $e_i \in g_e(x)$, then agents j and k are selected based on uniform random distributions. Consider agent i with $p_i = \ell$ and $x_i < 1$. Based on (13) and (14), the following probabilities of selecting agents j and k of a particular type hold. First, if $e_i \in E_1$, the probability that

(1a) agent j is not of type ℓ but agent k is equals

$$\begin{aligned} P[p_i \neq p_j, p_i = p_k | e_i \in E_1] &= P[p_i \neq p_j | e_i \in E_1]\, P[p_i = p_k | e_i \in E_1] \\ &= (1 - P[p_i = p_j | e_i \in E_1])\, P[p_i = p_k | e_i \in E_1] \\ &= \frac{n_\ell - qx_i - 1}{n - q - 1} \end{aligned}$$

(1b) agent j is of type ℓ and agent k is not equals

$$\begin{aligned} P[p_i = p_j, p_i \neq p_k | e_i \in E_1] &= P[p_i = p_j | e_i \in E_1]\, P[p_i \neq p_k | e_i \in E_1] \\ &= P[p_i = p_j | e_i \in E_1]\, (1 - P[p_i = p_k | e_i \in E_1]) \\ &= 0 \end{aligned}$$

(1c) agents j and k are both not of type ℓ equals

$$\begin{aligned} P[p_i \neq p_j, p_i \neq p_k | e_i \in E_1] &= (1 - P[p_i = p_j | e_i \in E_1])(1 - P[p_i = p_k | e_i \in E_1]) \\ &= 1 - \frac{n_\ell - qx_i - 1}{n - q - 1} \end{aligned}$$

(1d) agents j and k are both of type ℓ equals

$$\begin{aligned} P[p_i = p_j, p_i = p_k | e_i \in E_1] &= P[p_i = p_j | e_i \in E_1]\, P[p_i = p_k | e_i \in E_1] \\ &= 0 \end{aligned}$$

Second, based on (15) and (16), if $e_i \in E_2$, the probability that

(2a) agent j is not of type ℓ but agent k is equals

$$\begin{aligned} P[p_i \neq p_j, p_i = p_k | e_i \in E_2] &= P[p_i \neq p_j | e_i \in E_2]\, P[p_i = p_k | e_i \in E_2] \\ &= (1 - P[p_i = p_j | e_i \in E_2])\, P[p_i = p_k | e_i \in E_2] \\ &= 0 \end{aligned}$$

(2b) agent j is of type ℓ and agent k is not equals

$$\begin{aligned} P[p_i = p_j, p_i \neq p_k | e_i \in E_2] &= P[p_i = p_j | e_i \in E_2]\, P[p_i \neq p_k | e_i \in E_2] \\ &= P[p_i = p_j | e_i \in E_2]\, (1 - P[p_i = p_k | e_i \in E_2]) \\ &= x_i \end{aligned}$$

(2c) agents j and k are both not of type ℓ equals

$$\begin{aligned} P[p_i \neq p_j, p_i \neq p_k | e_i \in E_2] &= (1 - P[p_i = p_j | e_i \in E_2])(1 - P[p_i = p_k | e_i \in E_2]) \\ &= 1 - x_i \end{aligned}$$

(2d) agents j and k are both of type ℓ equals

$$\begin{aligned} P[p_i = p_j, p_i = p_k | e_i \in E_2] &= P[p_i = p_j | e_i \in E_2]\, P[p_i = p_k | e_i \in E_2] \\ &= 0 \end{aligned}$$

Finally, we can sum up the probabilities for both types of events. For $e_i \in g_e(x)$ such that $p_i = \ell$, the probability that

(a) agent j is not of type ℓ but agent k is (cases 1a and 2a) equals

$$\begin{aligned} P[p_i \neq p_j, p_i = p_k] &= P[p_i \neq p_j, p_i = p_k | e_i \in E_1]\, P[e_i \in E_1] \\ &\quad + P[p_i \neq p_j, p_i = p_k | e_i \in E_2]\, P[e_i \in E_2] \\ &= \varepsilon\, \frac{n_\ell - q x_i - 1}{n - q - 1} \end{aligned}$$

(b) agent j is of type ℓ and agent k is not (cases 1b and 2b) equals

$$\begin{aligned} P[p_i = p_j, p_i \neq p_k] &= P[p_i = p_j, p_i \neq p_k | e_i \in E_1]\, P[e_i \in E_1] \\ &\quad + P[p_i = p_j, p_i \neq p_k | e_i \in E_2]\, P[e_i \in E_2] \\ &= (1 - \varepsilon)\, x_i \end{aligned}$$

(c) agents j and k are both not of type ℓ (cases 1c and 2c) equals

$$\begin{aligned} P[p_i \neq p_j, p_i \neq p_k] &= P[p_i \neq p_j, p_i \neq p_k | e_i \in E_1]\, P[e_i \in E_1] \\ &\quad + P[p_i \neq p_j, p_i \neq p_k | e_i \in E_2]\, P[e_i \in E_2] \\ &= 1 - \varepsilon\, \frac{n_\ell - q x_i - 1}{n - q - 1} - (1 - \varepsilon)\, x_i \end{aligned}$$

(d) agents j and k are both of type ℓ (cases 1d and 2d) equals

$$\begin{aligned} P[p_i = p_j, p_i = p_k] &= P[p_i = p_j, p_i = p_k | e_i \in E_1]\ P[e_i \in E_1] \\ &\quad + P[p_i = p_j, p_i = p_k | e_i \in E_2]\ P[e_i \in E_2] \\ &= 0 \end{aligned}$$

If $x_i < 1$, then cases (a), (b) and (c) characterize the probability with which the transition operator f_e satisfies each conditional (see (3)). Next, consider agent i with $p_i = \ell$ and $x_i = 1$. Note that mechanism M1 cannot be triggered for agent i (because there is no neighboring agent of a different type), so

$$P[p_j \neq p_i, p_k = p_i] \ = \ P[p_j \neq p_i, p_k \neq p_i] \ = \ 0$$

Moreover, it is not possible that all three agents are of the same type, so

$$P[p_j = p_i, p_k = p_i] \ = \ 0$$

Therefore, if $x_i = 1$, then the probability that agents i and j are of the same type and agent k is not equals

$$P[p_j = p_i, p_k \neq p_i] = P[p_j = p_i, p_k \neq p_i | e_i \in E_2]\ P[e_i \in E_2] = (1-\varepsilon)\ x_i$$

Proof of Lemma 2. The probability that the cohesion index $w_u \in T_\ell$ decreases depends on the number of agents of type ℓ that can trigger events of type 2 (i.e., $i \in N_\ell$ such that $e_i \in E_2$). This probability equals the expected probability that an agent redirects a connection with an agent of the same type towards an agent of different type, that is

$$\bar{\pi}_u = \frac{1}{|\{i \in N_\ell : e_i \in E_2\}|} \sum_{i \in N_\ell : e_i \in E_2} P[p_i = p_j, p_i \neq p_k]$$

But because any agent can disconnect from a neighboring agent (thereby trigger mechanism M2), using (2) and (5), we get for agents of type ℓ the probability to transition to a lower cohesion index as

$$\bar{\pi}_u \ = \ \frac{1-\varepsilon}{n_\ell} \sum_{i \in N_\ell} x_i \ = \ (1-\varepsilon)\ w_u$$

On the other hand, the probability that the cohesion index $w_u \in T_\ell$ increases depends on the number of agents of type ℓ that can trigger events of type 1 (i.e., $i \in N_\ell$ such that $e_i \in E_1$). The expected probability that an agent redirects a connection with an agent of different type towards an agent of the same type is

$$\hat{\pi}_u = \frac{1}{|\{i \in N_\ell : e_i \in E_1\}|} \sum_{i \in N_\ell : e_i \in E_1} P[p_i \neq p_j, p_i = p_k] \tag{17}$$

Next, we consider worst-case combinations of the states of the agents to define lower and upper bounds on (17). In particular, for the lower bound on (17) we need to find the maximum number of agents that can trigger an event of type 1, which corresponds to the maximum number of agents with state below 1, that is

$$\max\{|\{i \in N_\ell : e_i \in E_1\}|\} = \max\{|\{i \in N_\ell : x_i < 1\}|\}$$

Note that events $e_i \in E_1$ cannot be triggered by agents with $x_i = 1$ because they do not have different type neighbors (i.e., $|Q_i'| = q$). The vector of all possible cohesion indices (defined by

(8)) can be written as

$$H_\ell = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_{c_\ell - n_\ell - 1} \\ w_{c_\ell - n_\ell} \\ w_{c_\ell - n_\ell + 1} \\ \vdots \\ w_{c_\ell - 1} \\ w_{c_\ell} \end{bmatrix} = \frac{1}{n_\ell} \begin{bmatrix} \frac{0}{q} \\ \frac{1}{q} \\ \vdots \\ \frac{(q-1)n_\ell - 1}{q} \\ \frac{(q-1)n_\ell}{q} \\ \frac{(q-1)n_\ell + 1}{q} \\ \vdots \\ \frac{qn_\ell - 1}{q} \\ \frac{qn_\ell}{q} \end{bmatrix} = \frac{1}{n_\ell} \begin{bmatrix} \frac{0}{q}(n_\ell) \\ \frac{0}{q}(n_\ell - 1) + \frac{1}{q}(1) \\ \vdots \\ \frac{q-2}{q}(1) + \frac{q-1}{q}(n_\ell - 1) \\ \frac{q-1}{q}(n_\ell) \\ \frac{q-1}{q}(n_\ell - 1) + \frac{q}{q}(1) \\ \vdots \\ \frac{q-1}{q}(1) + \frac{q}{q}(n_\ell - 1) \\ \frac{q}{q}(n_\ell) \end{bmatrix}$$

Note that in the worst-case scenario, for $u \in \{1, \dots, c_\ell - n_\ell\}$, n_ℓ agents have $x_i < 1$ because

$$\sum_{i \in N_\ell} x_i \leq \frac{q-1}{q}\, n_\ell$$

For $u \in \{c_\ell - n_\ell + 1, \dots, c_\ell\}$, $c_\ell - u$ agents have $x_i = (q-1)/q < 1$ because

$$\frac{q-1}{q}\, n_\ell < \sum_{i \in N_\ell} x_i = \frac{q-1}{q}(c_\ell - u) + q(n_\ell - c_\ell + u)$$

The maximum number of agents $i \in N_\ell$ such that $e_i \in E_1$ is given by

$$\begin{aligned} \max\{|\{i \in N_\ell : e_i \in E_1\}|\} &= \begin{cases} n_\ell, & \text{if } u \in \{2, \dots, c_\ell - n_\ell\}; \\ c_\ell - u, & \text{if } u \in \{c_\ell - n_\ell + 1, \dots, c_\ell - 1\}. \end{cases} \\ &= \begin{cases} n_\ell, & \text{if } u \in \{2, \dots, c_\ell - n_\ell - 1\}; \\ c_\ell - u, & \text{if } u \in \{c_\ell - n_\ell, \dots, c_\ell - 1\}. \end{cases} \end{aligned} \tag{18}$$

According to (17) and (18), we know that if $u \in \{2, \dots, c_\ell - n_\ell - 1\}$

$$\hat{\pi}_u \geq \frac{1}{n_\ell} \sum_{i \in N_\ell} P[p_i \neq p_j, p_i = p_k]$$

Using (2) and (4), the previous inequality can be written as

$$\begin{aligned} \hat{\pi}_u &\geq \frac{\varepsilon}{n_\ell} \sum_{i \in N_\ell} \frac{n_\ell - qx_i - 1}{n - q - 1} \\ &= \frac{\varepsilon}{n - q - 1} \left(\frac{1}{n_\ell} \sum_{i \in N_\ell} (n_\ell - 1) - \frac{q}{n_\ell} \sum_{i \in N_\ell} x_i \right) \\ &= \frac{\varepsilon}{n - q - 1} \left(n_\ell - 1 - \frac{q}{n_\ell} \sum_{i \in N_\ell} x_i \right) \\ &= \varepsilon\, \frac{n_\ell - qw_u - 1}{n - q - 1} \end{aligned}$$

Similarly, if $u \in \{c_\ell - n_\ell, \ldots, c_\ell - 1\}$

$$
\begin{aligned}
\hat{\pi}_u &\geq \frac{1}{c_\ell - u} \sum_{i \in N_\ell : e_i \in E_1} P[p_i \neq p_j, p_i = p_k] \\
&= \frac{\varepsilon}{c_\ell - u} \sum_{i \in N_\ell : e_i \in E_1} \frac{n_\ell - q x_i - 1}{n - q - 1} \\
&= \frac{\varepsilon}{n - q - 1} \left(\frac{1}{c_\ell - u} \sum_{i \in N_\ell : e_i \in E_1} (n_\ell - 1) - \frac{q}{c_\ell - u} \sum_{i \in N_\ell : e_i \in E_1} x_i \right) \\
&= \frac{\varepsilon}{n - q - 1} \left(n_\ell - 1 - q\, \frac{q - 1}{q} \right) \\
&= \varepsilon\, \frac{n_\ell - q}{n - q - 1}
\end{aligned}
$$

For the upper bound on (17) we need to consider the minimum number of agents that can trigger an event of type 1, which corresponds to the minimum number of agents with state below 1, that is

$$
\min\{|\{i \in N_\ell : e_i \in E_1\}|\} = \min\{|\{i \in N_\ell : x_i < 1\}|\}
$$

Again for the worst-case scenario, the vector of all possible cohesion indices (defined by (8)) can be written as

$$
H_\ell = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_q \\ w_{q+1} \\ w_{q+2} \\ \vdots \\ w_{c_\ell - q - 1} \\ w_{c_\ell - q} \\ w_{c_\ell - q + 1} \\ \vdots \\ w_{c_\ell - 1} \\ w_{c_\ell} \end{bmatrix} = \frac{1}{n_\ell} \begin{bmatrix} \frac{0}{q} \\ \frac{1}{q} \\ \vdots \\ \frac{q-1}{q} \\ \frac{q}{q} \\ \frac{q+1}{q} \\ \vdots \\ \frac{q(n_\ell - 1) - 1}{q} \\ \frac{q(n_\ell - 1)}{q} \\ \frac{q(n_\ell - 1) + 1}{q} \\ \vdots \\ \frac{q n_\ell - 1}{q} \\ \frac{q n_\ell}{q} \end{bmatrix} = \frac{1}{n_\ell} \begin{bmatrix} \frac{0}{q}(n_\ell) \\ \frac{0}{q}(n_\ell - 1) + \frac{1}{q}(1) \\ \vdots \\ \frac{0}{q}(n_\ell - 1) + \frac{q-1}{q}(1) \\ \frac{0}{q}(n_\ell - 1) + \frac{q}{q}(1) \\ \frac{0}{q}(n_\ell - 2) + \frac{1}{q}(1) + \frac{q}{q}(1) \\ \vdots \\ \frac{0}{q}(1) + \frac{q-1}{q}(1) + \frac{q}{q}(n_\ell - 2) \\ \frac{0}{q}(1) + \frac{q}{q}(n_\ell - 1) \\ \frac{1}{q}(1) + \frac{q}{q}(n_\ell - 1) \\ \vdots \\ \frac{q-1}{q}(1) + \frac{q}{q}(n_\ell - 1) \\ \frac{q}{q}(n_\ell) \end{bmatrix}
$$

Note that for $u \in \{qr + 1, ..., q(r + 1)\}$, $r \in \{0, \ldots, n_\ell - 1\}$, $n_\ell - r$ agents have $x_i < 1$ because

$$
\frac{qr}{q} \leq \sum_{i \in N_\ell} x_i \leq \frac{(q - 1) + qr}{q}
$$

For $u = c_\ell$, no agents have $x_i < 1$ because

$$
\sum_{i \in N_\ell} x_i = \frac{q n_\ell}{q}
$$

So, the minimum number of agents $i \in N_\ell$ such that $e_i \in E_1$ is

$$\min\{|\{i \in N_\ell : e_i \in E_1\}|\} = \begin{cases} n_\ell, & \text{if } u \in \{2, ..., q\}; \\ n_\ell - r, & \text{if } u \in \{qr+1, ..., q(r+1)\} \\ & \text{and } r \in \{1, \dots, n_\ell - 1\}. \end{cases} \tag{19}$$

Because the probability in (4) reaches the maximum at $x_i = 0$, we know that if every agent in the set $\{i \in N_\ell : e_i \in E_1\}$ has $x_i = 0$, then the sum on the right of (17) reaches the maximum value. The cohesion indices satisfying this condition are w_{qr+1}, $r \in \{1, \dots, n_\ell - 1\}$. Therefore, based on (4), (17), and (19)

$$\begin{aligned} \hat{\pi}_u &\leq \frac{\varepsilon}{\min\{|\{i \in N_\ell : e_i \in E_1\}|\}} \sum_{i \in N_\ell : e_i \in E_1} \frac{n_\ell - 1}{n - q - 1} \\ &= \varepsilon \, \frac{n_\ell - 1}{n - q - 1} \end{aligned}$$

Proof of Theorem 1. From the transition diagram in Fig. 2, notice that starting at w_i, $i \in \{2, \dots, c_\ell - 1\}$ the probability of reaching a cohesion index w_{i-1} (w_{i+1} respectively) is equal to $\bar{\pi}_i$ ($\hat{\pi}_i$). The probability that the group maintains the same cohesion index is $1 - \hat{\pi}_i - \bar{\pi}_i$. The probability of reaching w_1 rather than w_{c_ℓ} when starting at w_i is

$$\begin{aligned} \alpha_i &= \bar{\pi}_i \alpha_{i-1} + (1 - \hat{\pi}_i - \bar{\pi}_i)\alpha_i + \hat{\pi}_i \alpha_{i+1} \\ &= \frac{\bar{\pi}_i}{\hat{\pi}_i + \bar{\pi}_i}\alpha_{i-1} + \frac{\hat{\pi}_i}{\hat{\pi}_i + \bar{\pi}_i}\alpha_{i+1} \\ &= \frac{\bar{\pi}_i}{\hat{\pi}_i + \bar{\pi}_i}\alpha_{i-1} + \left(1 - \frac{\bar{\pi}_i}{\hat{\pi}_i + \bar{\pi}_i}\right)\alpha_{i+1} \end{aligned}$$

Subtracting the term $(1 - \bar{\pi}_i/(\hat{\pi}_i + \bar{\pi}_i))\,\alpha_{i1}$ on both sides of the previous equation

$$\alpha_i - \left(1 - \frac{\bar{\pi}_i}{\hat{\pi}_i + \bar{\pi}_i}\right)\alpha_i = \frac{\bar{\pi}_i}{\hat{\pi}_i + \bar{\pi}_i}\alpha_{i-1} + \left(1 - \frac{\bar{\pi}_i}{\hat{\pi}_i + \bar{\pi}_i}\right)\alpha_{i+1} - \left(1 - \frac{\bar{\pi}_i}{\hat{\pi}_i + \bar{\pi}_i}\right)\alpha_i$$

and we have

$$\begin{aligned} \frac{\bar{\pi}_i}{\hat{\pi}_i + \bar{\pi}_i}\Big(\alpha_i - \alpha_{i-1}\Big) &= \left(1 - \frac{\bar{\pi}_i}{\hat{\pi}_i + \bar{\pi}_i}\right)\Big(\alpha_{i+1} - \alpha_i\Big) \\ &= \frac{\hat{\pi}_i}{\hat{\pi}_i + \bar{\pi}_i}\Big(\alpha_{i+1} - \alpha_i\Big) \end{aligned}$$

Therefore

$$\alpha_{i+1} - \alpha_i = \frac{\bar{\pi}_i}{\hat{\pi}_i}\Big(\alpha_i - \alpha_{i-1}\Big)$$

For all $u \in \{2, \dots, c_\ell - 1\}$, the previous equation can be written as

$$\begin{aligned} \alpha_{i+1} - \alpha_i &= \frac{\bar{\pi}_i}{\hat{\pi}_i}\Big(\alpha_i - \alpha_{i-1}\Big) \\ &= \frac{\bar{\pi}_i}{\hat{\pi}_i}\left(\frac{\bar{\pi}_{i-1}}{\hat{\pi}_{i-1}}\Big(\alpha_{i-1} - \alpha_{i-2}\Big)\right) \\ &\quad \vdots \\ &= \prod_{j=u}^{i} \frac{\bar{\pi}_j}{\hat{\pi}_j}(\alpha_u - \alpha_{u-1}) \end{aligned}$$

Summing over $i = u, \ldots, c_\ell - 1$, we get

$$\sum_{i=u}^{c_\ell-1} \Big(\alpha_{i+1} - \alpha_i\Big) = (\alpha_u - \alpha_{u-1}) \sum_{i=u}^{c_\ell-1} \prod_{j=u}^{i} \frac{\bar{\pi}_j}{\hat{\pi}_j}$$

which yields

$$\alpha_{c_\ell} - \alpha_u = (\alpha_u - \alpha_{u-1}) \sum_{i=u}^{c_\ell-1} \prod_{j=u}^{i} \frac{\bar{\pi}_j}{\hat{\pi}_j}$$

Since $\alpha_{c_\ell} = 0$ and $\alpha_1 = 1$ we obtain for $u \in \{2, \ldots, c_\ell - 1\}$

$$\begin{aligned}
\alpha_u &= \frac{\displaystyle\sum_{i=u}^{c_\ell-1} \prod_{j=u}^{i} \frac{\bar{\pi}_j}{\hat{\pi}_j}}{\displaystyle 1 + \sum_{i=u}^{c_\ell-1} \prod_{j=u}^{i} \frac{\bar{\pi}_j}{\hat{\pi}_j}} \, \alpha_{u-1} \\
&= \left(\frac{\displaystyle\sum_{i=u}^{c_\ell-1} \prod_{j=u}^{i} \frac{\bar{\pi}_j}{\hat{\pi}_j}}{\displaystyle 1 + \sum_{i=u}^{c_\ell-1} \prod_{j=u}^{i} \frac{\bar{\pi}_j}{\hat{\pi}_j}} \right) \left(\frac{\displaystyle\sum_{i=u-1}^{c_\ell-1} \prod_{j=u-1}^{i} \frac{\bar{\pi}_j}{\hat{\pi}_j}}{\displaystyle 1 + \sum_{i=u-1}^{c_\ell-1} \prod_{j=u-1}^{i} \frac{\bar{\pi}_j}{\hat{\pi}_j}} \right) \cdots \left(\frac{\displaystyle\sum_{i=2}^{c_\ell-1} \prod_{j=2}^{i} \frac{\bar{\pi}_j}{\hat{\pi}_j}}{\displaystyle 1 + \sum_{i=2}^{c_\ell-1} \prod_{j=2}^{i} \frac{\bar{\pi}_j}{\hat{\pi}_j}} \right) \\
&= \prod_{k=2}^{u} \frac{\displaystyle\sum_{i=k}^{c_\ell-1} \prod_{j=k}^{i} \frac{\bar{\pi}_j}{\hat{\pi}_j}}{\displaystyle 1 + \sum_{i=k}^{c_\ell-1} \prod_{j=k}^{i} \frac{\bar{\pi}_j}{\hat{\pi}_j}} \\
&= \prod_{k=2}^{u} \left(1 - \frac{1}{\displaystyle 1 + \sum_{i=k}^{c_\ell-1} \prod_{j=k}^{i} \frac{\bar{\pi}_j}{\hat{\pi}_j}} \right)
\end{aligned}$$